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M-theory compactifications on certain 'toric' cones of G_2 holonomy

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Abstract: We develop methods to study the singularities of certain G_2 cones related to toric hyperkähler spaces and Einstein selfdual orbifolds. This allows us to determine the low energy gauge groups of chiral $N = 1$ compactifications of M-theory on a large family of such backgrounds, which includes the models recently studied by Acharya and Witten. All M-theory compactifications belonging to our family admit a T^2 of isometries, and therefore T-dual IIA and IIB descriptions. We argue that reduction through such an isometry leads generically to systems of weakly and strongly coupled IIA 6-branes, T-dual to delocalized type-IIB 5-branes. We find a simple criterion for the existence of a 'good' isometry which leads to IIA models containing only weakly-coupled D6-branes (up to possible unknown effects encoded by a CFT in 5 dimensions), and construct examples of such backgrounds.

Keywords: M-Theory, D-branes, Differential and Algebraic Geometry.

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1. Introduction

Compactifications of M-theory on spaces of G_2 holonomy have recently attracted renewed attention (see [54, 52, 50, 14, 7, 2, 4, 3, 8, 7] for a very partial list of references). Perhaps the most important discovery from a physical perspective is that M-theory compactifications on certain singular G_2 spaces can lead to $N = 1$ supersymmetric nonabelian gauge theories in four dimensions with chiral matter [14, 50]. This contrasts markedly with the smooth case, which upon compactification leads to abelian vector multiplets and no net chirality [9]. The appearance of chiral matter in the singular case (more precisely, for G_2 spaces with singularities in codimension four) can be seen indirectly through an anomaly cancellation argument [14]. It can also be demonstrated in certain particular cases by reducing a local description of such singularities to a system of D6-branes in type-IIA string theory [50] (see also [5, 6]).

Since chiral nonabelian gauge theories are of obvious relevance to phenomenology, such compactifications may offer a way to study the strong coupling limit of various models whose physics has traditionally been accessible only at weak coupling. Unfortunately, progress in this direction is obstructed by our very poor understanding of G_2-holonomy spaces, which makes it very difficult to extract specific results. In fact, current knowledge provides extremely few constructions of G_2 spaces, both in the compact and non-compact set-up. Among these are compact examples of the type $(CY \times S^1)/\mathbb{Z}_2$ due to [11, 12] as well as two other constructions due to [11] and [13]. In the non-compact case, our knowledge seems to be largely limited to certain G_2 lifts of the local geometry of the conifold [2, 3, 4, 7] (which are known for the case of a 'simple' conifold point, i.e. a point where a single rational curve or a 3-sphere is collapsed) as well as a construction of conical G_2 spaces due to Bryant and Salamon [37] and Gibbons, Page and Pope [38]. The latter allows one to produce an infinite family of G_2 cones. The construction of [37, 38] starts with a four-dimensional, Einstein-self-dual space M of positive scalar curvature and builds a G_2 metric on the real cone $\mathcal{C}(Y)$ over its twistor space Y. In the case considered in [37, 38] (namely when M is smooth), this leads to only two conical G_2 metrics, since a smooth Einstein selfdual four-manifold of positive scalar curvature must coincide with either of S^4 or $\mathbb{CP}^2$. M-theory physics on the G_2 cones obtained from these two choices was studied in [52]. However, the construction of [37, 38] generalizes to the case when M is allowed to have orbifold singularities. As is by now well-known [36, 24, 16, 15, 28, 27], compact Einstein self-dual orbifolds of positive scalar curvature admit a rich geometry, and in particular there exists an infinity of inequivalent examples of such spaces. A very small class among these is given by the celebrated models of [36], which are orbifold equivalent with the weighted projective spaces $W\mathbb{CP}^2_{p,q,r}$, but endowed with an Einstein self-dual metric (which differs from the usual (toric) metric). These models are very special in many regards. For example, it is shown in [16] that the orbifold ESD metrics of [36] are hermitean, and therefore conformal to certain Bochner-Kähler metrics [15, 10].

In the paper [50], the G_2 cone construction of [37, 38] was applied to the ESD orbifolds M of [36] and used to show that M-theory compactification on the associated G_2 cone leads to chiral nonabelian gauge theories in four dimensions. The analysis of [50] relies on knowledge of the singularity structure of $M = W\mathbb{CP}^2_{p,q,r}$, which is used in order to extract the singularities of Y and thus of $\mathcal{C}(Y)$. The location and type of such singularities is crucial for the physical analysis, since they allow one to identify the resulting low energy gauge group as well as (part of) the associated type-IIA description. For particular values of p, q, r, the authors of [50] show that performing the Kaluza-Klein reduction with respect to an appropriately chosen isometry of the G_2 metric leads to a system of three D6-branes in type-IIA string theory; this gives an alternate explanation for the presence of chiral fermions.

Since the choice of ESD orbifold M used by [50] is very restrictive, this particular class of models does not allow for the construction of G_2 lifts of more generic D-brane systems. To extend it, one can use the construction of [37, 38] with a more general choice for the ESD orbifold M. The purpose of the present paper is to carry out part of this generalization, namely by considering ESD orbifolds M which admit a two-torus of isometries.

While the most general ESD metric admitting two independent and commuting isometries is explicitly known due to the work of Calderbank and Pedersen[1] [24], its expression is rather complicated and a direct analysis of singularities starting from the metric is quite involved. This makes it difficult to apply the simple methods of [50] to more general examples. Instead, we shall proceed in indirect fashion, by using a now well-known correspondence [28, 27, 32, 51, 17, 18] between a quaternion-Kähler orbifold M, its twistor space Y and its *hyperkähler cone* X.[2] This allows us to determine the singularities of $\mathcal{C}(Y)$ by studying the singularities of the hyperkähler cone X and the fixed points of a certain U(1) action on X whose Kähler quotient recovers the twistor space. To make the analysis manageable, we shall require that the cone X be *toric hyperkähler* [34], i.e. a hyperkähler quotient (at zero moment map level) of some affine quaternion space by a torus action. This amounts to requiring the existence of two commuting, independent and 'compact' isometries of M and will allow us to use a slight adaptation of the results of [34] in order to identify[3] the singularities of X; the singularities of Y then follow from an analysis of the U(1) reduction. In fact, we will be able to extract a simple algorithm for identifying such singularities, and therefore the low energy gauge group of M-theory on such G_2 holonomy backgrounds.

The G_2 cones considered in this paper possess a two-torus of isometries, induced by those of the Einstein self-dual space M. Upon choosing one of these, one can perform the associated IIA reduction, thereby producing a certain configuration of D6-branes. The remaining isometries of the G_2 cone descend to an S^1 of isometries of the IIA metric, which allows one to obtain a T-dual IIB description. The nature of the resulting IIA and IIB solutions depends on the choice of isometry used to reduce the G_2 metric.

By characterizing such isometries in toric hyperkähler language, we argue that one will generically obtain a system of strongly coupled 6-branes in IIA and a T-dual system of strongly coupled, delocalized 5-branes in IIB. By varying the choice of the isometry used to perform the reduction, one is generally able to bring some of these branes to weak coupling. Whether a weakly-coupled description can be achieved for all D-branes is a characteristic of the model, and depends on the existence of a 'good' isometry, a property which can be tested through a simple criterion. In particular, we are able to construct models which

[1] Particular metrics of this type were considered for example in [25] and [26].

[2] This can be completed by including a fourth geometry, namely a 3-Sasaki space [27, 28] — as we recall in section 3.

[3] It will turn out that a nice description can be obtained if one imposes a further technical condition, namely that X be a 'good' toric hyperkähler cone (see sections 2 and 5).

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do admit such an isometry, and therefore lead to IIA and IIB descriptions containing no strongly-coupled branes. Unfortunately, the complete physics of these particular examples is not entirely encoded in the D-brane data, because the associated G_2 cones turn out to also contain singularities in codimension six. As argued in [50], the extra-physics induced by such singularities should be governed by a conformal field theory in five dimensions, whose precise description remains unknown.[4]

It will be clear from our description that models with a good isometry are rather rare, and in particular that the generic models produced by our construction do not allow for an interpretation only in terms of weakly coupled branes. Among the models which admit a good isometry, we shall find some whose type-IIA description is given in terms of three D6-branes, carrying the same nonabelian gauge groups as those obtained from certain models considered in [50]. Since these are realized through different geometries, they must correspond to different choices of relative angles between the branes, up to possible effects governed by a 5-dimensional conformal field theory, which we will not able to identify.

The present paper is organized as follows. In section 2, we give a short summary of our main results and illustrate them with one of the models already analyzed in [50] with different methods. section 3 recalls the construction of [37] and [38], as well as the relation between ESD spaces, twistor spaces, toric hyperkähler cones and 3-Sasaki spaces. It also discusses certain isometries of the G_2 cone of [37, 38], which can be traced back to triholomorphic isometries of the hyperkähler cone of M. Section 4 introduces toric hyperkähler spaces following [34], but under slightly more general assumptions. Section 5 discusses toric hyperkähler cones. After giving their description as an intersection of quadrics in a toric ambient space, we discuss a presentation of such cones as torus fibrations over a real affine space, which will play an important role in the remainder of the paper. We also discuss the hyperkähler potential of such spaces, and give a criterion for identifying their singularities (this extends certain results of [34] to the case of hyperkähler cones). Much of the discussion of section 5 is valid for $\dim X = 4n$ with arbitrary n, and we present it in this generality due to its relevance for other problems, such as the study of hypermultiplet moduli spaces in $N = 2$ supergravity in four dimensions. Subsection 5.4.3. specializes our results to the eight-dimensional case, which is relevant for the remainder of the paper. In section 6, we apply the general construction reviewed in section 3 to the case of toric hyperkähler cones. This allows us to describe the associated twistor space, quaternion-Kähler space and 3-Sasaki space in terms of certain explicit quotients. We also extract a description of the twistor space as an intersection of quadrics in a toric variety, and present a criterion for effectiveness of a certain U(1) action on X which leads to the twistor space

[4]Using the criteria extracted in the present paper, we performed a computer search for cones admitting a good isometry but no codimension six singularities. Our search included a range of 6000 models whose hyperkähler cone is given respectively by a $\mathrm{U}(1)^2$ or $\mathrm{U}(1)^3$ quotient of $\mathbf{H}^3$ or $\mathbf{H}^4$. This limited search produced no new examples satisfying *both* constraints. This suggests that the class of G_2 cones considered in this paper does not produce such 'very good' models beyond the examples $W_{p,p,q}$ studied by Acharya and Witten. However, we do not have a proof of this fact, and our failure to find such examples may simply be due to the limits of our computer search. By comparison, it is rather straightforward to produce large classes of models admitting a good isometry, but containing some codimension six singularities, and we shall present some explicit examples in section 8.

upon performing the Kähler reduction. In section 7, we use these preparations in order to extract the location and type of singularities of the twistor space Y, which immediately determine the singularities of the G_2 cones $\mathcal{C}(Y)$. We start by showing that all singularities of Y must lie on a certain *distinguished locus*, which is naturally associated with a *characteristic polygon*. After presenting criteria for identifying the singularities of Y, we discuss the issue of good isometries. Finally, we mention an alternate, toric geometry approach to finding the singularities of Y. In section 8, we apply the results of section 7 to a few explicit models. This allows us to extract the associated low energy gauge groups and discuss the existence of good isometries. In particular, we present new models which admit such an isometry. Section 9 discusses M-theory reduction to type IIA on our backgrounds, as well as its T-dual, type-IIB description. By using geometric arguments, we extract the interpretation of such models in terms of (generally strongly coupled) branes. Section 10 presents our conclusions. Some technical results are discussed in appendices A–E.

2. Summary of results

As mentioned in the introduction, the singularities of the G_2 cone $\mathcal{C}(Y)$ are immediately determined by the singularities of the twistor space Y. Recall that Y is an S^2 fibration over a four-dimensional, Einstein selfdual space M. The twistor space is a compact, 3-dimensional complex variety, which turns out to have singularities both on points and along certain holomorphically embedded two-spheres. In algebraic geometry language, the later are compound du Val singularities [41], in our case families of A_n singularities depending on one complex parameter. In our models, there will be two types of spheres which may become singular:

(*a*) Vertical spheres, i.e. spheres which are fibers of $Y \to M$

(*b*) Horizontal spheres, which are lifts of spheres lying in M.

As explained in the introduction, the twistor space can be viewed as a Kähler reduction $Y = X//_{\zeta}\,\mathrm{U}(1)$ of a hyperkähler cone X at some positive moment map level ζ. Since the action of $\mathrm{U}(1)$ on X is uniquely determined, Y is completely specified by the choice of X.

The results of the present paper concern the particular case when X is a *toric hyperkähler cone*, which means that it can be presented as a *hyperkähler* reduction (at zero level) of some affine quaternion space $\mathbf{H}^n$ through the abelian group $T^{n-2} = \mathrm{U}(1)^{n-2}$:

$$X = \mathbf{H}^n///_0 T^{n-2}\,. \tag{2.1}$$

Such a quotient is completely specified by the choice of $\mathrm{U}(1)^{n-2}$ action, which we characterize by the associated charge matrix Q. This is an $(n-2)\times n$ matrix whose entries $Q_{\alpha j} = q_j^{(\alpha)}$ appear in the transformation rules of the quaternion coordinates $u_1, \ldots, u_n$ of $\mathbf{H}^n$:

$$u_k \to \prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_k^{(\alpha)}} u_k \qquad (k = 1, \ldots, n)\,, \tag{2.2}$$

with λ_α some complex numbers of unit modulus. Note that (2.2) maps $\mathrm{U}(1)^{n-2}$ into the torus $T^n = \mathrm{U}(1)^n$ which acts diagonally on u_k:

$$u_k \to \Lambda_k u_k \qquad (\Lambda_k \in \mathrm{U}(1))\,. \tag{2.3}$$

We shall impose two technical conditions on the matrix Q:

(A) We require that Q is 'torsion-free', i.e. that the greatest common divisor of all of its $(n-2) \times (n-2)$ minor determinants equals one. Equivalently, the integral Smith form[5] of Q is $[I_{n-2}, 0]$, where I_{n-2} is the $(n-2)\times(n-2)$ identity matrix (in particular, Q has maximal rank, so that its rows are linearly independent over the reals). This condition assures that the action of $\mathrm{U}(1)^{n-2}$ on $\mathbf{H}^n$ is effective.

(B) We also require that the matrix Q is *good*, which means that none of its $(n-2)\times(n-2)$ minors can vanish.

If X is such a hyperkähler cone, then one can describe the points of the twistor space as follows. Picking the first complex structure of $\mathbf{H}^n$, we decompose u_k into complex coordinates:

$$u_k = w_k^{(+)} + \mathbf{j} w_k^{(-)}\,. \tag{2.4}$$

Then a point of Y is specified by $\{w_k^{(\pm)}\}$, taken to obey certain moment map constraints and considered modulo the action (2.2):

$$w_k^{(\pm)} \to \prod_{\alpha=1}^{n-2} \lambda_\alpha^{\pm q_k^{(\alpha)}} w_k^{(\pm)} \tag{2.5}$$

and the action of the 'projectivising' U(1) group by which one quotients to produce Y:

$$w_k^{(\pm)} \to \lambda w_k^{(\pm)}\,. \tag{2.6}$$

Let us define a $2 \times n$ integral matrix G as the 'transpose of the kernel of Q'. More precisely, the rows of G are *integral* and *primitive*[6] vectors which form a basis for the real vector space $\ker Q$. The columns of G are two-dimensional integral vectors $\nu_1, \ldots, \nu_n$, which we shall call *toric hyperkähler generators*. Note that we do *not* require that ν_j be primitive. It will also be convenient to consider the $(n-1)\times(2n)$ matrix $\tilde{Q} = \left[\begin{smallmatrix} Q & & -Q \\ 1 & \ldots & 1 \end{smallmatrix}\right]$.

With the hypotheses (A), (B), we prove the following results:

[5]Given an integral $r\times n$ matrix F with $r \leq n$, one can find matrices $U \in GL(r, \mathbb{Z})$ and $V \in GL(n, \mathbb{Z})$ such that the matrix $F^{\mathrm{ismith}} = U^{-1}FV$ has the *integral Smith form* $F_{\mathrm{ismith}} = [D, 0]$, where $D = \mathrm{diag}(t_1 \ldots t_r)$, with $t_1, \ldots, t_r$ some non-negative integers satisfying the division relations $t_1|t_2|\ldots|t_r$. These integers are called the *invariant factors* (or 'torsion coefficients') of F, and their product $t_1, \ldots, t_r$ coincides with the r^{th} discriminantal divisor $\mathfrak{g}(F)$ of F, which is the greatest common divisor of all $r \times r$ minors of F. It is clear that $\mathfrak{g}(F) = 1$ if and only if all t_i equal one. A similar result holds for $r > n$.

[6]We remind the reader that an integral vector is called primitive if the greatest common divisor of its components equals one.

(0a) A point u of the hyperkähler cone X can be singular only if one of the quaternion coordinates u_j vanishes. If two quaternion coordinates vanish, then u must coincide with the apex of X (i.e. one has $u = 0$). In particular, all singularities of X must occur on one of the four-dimensional loci X_j defined by the equations $u_j = 0$, and two such loci intersect at a single point, namely the apex of X.

Suppose that u is such that $u_j = 0$ but $u \neq 0$. Then u is a singular point of X if and only if the associated toric hyperkähler generator ν_j fails to be primitive. In this case, X has a $\mathbb{Z}_{m_j}$ singularity at u, where m_j is the greatest common divisor of the coordinates of ν_j.

(0b) The trivially acting subgroup of the projectivising U(1) is the trivial group or the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$. The $\mathbb{Z}_2$ subgroup acts trivially on X if and only if there exists a collection of rows of Q whose sum is a vector all of whose entries are odd. Such a collection of rows is unique if it exists.

1. If u is a singular point of the twistor space Y, then one of the following holds:

(1a) $u_j = 0$ (i.e. $w_j^{(+)} = w_j^{(-)} = 0$) for some $j = 1, \ldots, n$

or

(1b) There exists a choice of signs $\epsilon_j = \pm 1$ such that $w_1^{(-\epsilon_1)} = w_2^{(-\epsilon_2)} = \cdots = w_n^{(-\epsilon_n)} = 0$.

Condition (1a) defines a locus Y_j in Y, while (1b) defines a locus Y_ϵ. The union Y_H of all Y_ϵ will be called *the horizontal locus*, while the union Y_V of all Y_j is the *vertical locus*. The union $Y_D := Y_H \cup Y_V$ will be called the *distinguished locus*. We note that some horizontal components Y_ϵ may be void, and that some of the components Y_j, Y_ϵ may consist of smooth points of Y.

2. Every component Y_j is a vertical sphere of Y. A component Y_ϵ is either void or a horizontal sphere (as defined on page 6).

3. The planar polygon $\Delta \subset \mathbb{R}^2$ defined by:

$$\Delta = \left\{ a \in \mathbb{R}^2 \mid \sum_{j=1}^{n} |\nu_j \cdot a| = \zeta \right\} \tag{2.7}$$

will be called the *characteristic polygon*. It has the following properties:

(3a) Δ is convex polygon on $2n$ vertices and is symmetric with respect to the sign inversion $a \to -a$ of the plane. Moreover, its principal diagonals D_j (i.e. those diagonals of Δ which pass through the origin) lie on the lines h_j defined by $\nu_j \cdot a = 0$.

(3b) Every two-sphere Y_j of the vertical locus can be written as an S^1 fibration over the principal diagonal D_j of Δ; the S^1 fiber degenerates to a point above the opposite vertices of Δ connected by this diagonal.

(3c) Every *non-void* component of the horizontal locus is associated with an edge of Δ. If e is such an edge, then the associated component of Y_H is $Y_e := Y_{\epsilon(e)}$, where $\epsilon(e)$ is the collection of signs defined by:

$$\epsilon_k(e) = \mathrm{sign}(\nu_k \cdot p_e) \qquad (k = 1, \ldots, n)\,, \tag{2.8}$$

with p_e the vector from the origin to the middle point of the edge e. Any component Y_ϵ of Y_H for which ϵ cannot be written in this form must be void. In particular, Y_H contains precisely $2n$ non-void components Y_e.

(3d) If e is an edge of Δ, then Y_e is an S^1 fibration over e, whose fiber degenerates to a point at the two vertices of Δ connected by e.

(3e) Each vertex A of Δ corresponds to a common point Y_A of two horizontal spheres Y_e and $Y_{e'}$ and one vertical sphere Y_j, where e, e' are the edges of Δ adjacent to the given vertex and D_j is the principal diagonal passing through it.

(3f) The antipodal map acting on the fibers of $Y \to M$ covers the sign inversion $\iota : a \to -a$ of Δ when restricted to the horizontal locus Y_H. This map takes each horizontal sphere Y_e into the sphere Y_{-e} associated with the opposite edge while preserving the vertical components Y_j.

(3g) The ESD space M is topologically a T^2 fibration over the compact planar convex polytope bounded by the polygon Δ_M which results from Δ upon dividing through the sign inversion ι. The T^2 fibers of $M \to \Delta_M$ degenerate to circles above its edges and to points above its vertices.

4. Assume that the $\mathbb{Z}_2$ subgroup of the projectivising U(1) acts non-trivially on X (see criterion (0b)). Then the singularity type of Y along Y_j and Y_e can be determined as follows:

(4a) Given a horizontal sphere Y_e, consider the integral vector:

$$\nu_e = \sum_{k=1}^{n} \epsilon_k(e)\nu_k\,. \tag{2.9}$$

(It can be shown that this vector cannot vanish). Let m_e be the greatest common divisor of the two components of ν_e. Then Y has a $\Gamma_e = \mathbb{Z}_{m_e}$ singularity at every point of Y_e, possibly with the exception of the two points lying above the vertices connected by the edge e. The generator of $\mathbb{Z}_{m_e}$ acts as follows on the complex coordinates $w_k^{(-\epsilon_k(e))}$ transverse to the locus Y_e:

$$w_k^{(-\epsilon_k(e))} \to e^{\frac{2\pi i}{m_e}}\, w_k^{(-\epsilon_k(e))}\,. \tag{2.10}$$

The horizontal spheres Y_e and Y_{-e} associated with opposite edges have the same singularity type (since $\epsilon_k(-e) = -\epsilon_k(e)$).

(4b) a vertical sphere Y_j, consider the matrix $\tilde{Q}_j$ obtained by deleting the j^{th} and $(j+n)^{\mathrm{th}}$ columns of $\tilde{Q}$. Then singularity group Γ_j of Y along Y_j coincides with

$\mathbb{Z}_{m_j}$ or $\mathbb{Z}_{2m_j}$. To find which of these cases occurs, one computes the integral Smith form of the matrix $\tilde{Q}_j$, which is assured to be of the type:

$$\tilde{Q}_j^{\text{ismith}} = [\text{diag}(1\dots 1, t_j), 0]\,, \tag{2.11}$$

where $t_j = m_j$ or $t_j = 2m_j$. The singularity group Γ_j coincides with $\mathbb{Z}_{t_j}$. The generator of $\mathbb{Z}_{t_j}$ acts on the transverse coordinates as:

$$u_j \to e^{\frac{2\pi i}{t_j}} u_j \Leftrightarrow w_j^{(\pm)} \to e^{\pm\frac{2\pi i}{t_j}} w_j^{(\pm)}\,. \tag{2.12}$$

(4c) Given a vertex A of Δ, let e, e' and D_j be the two edges and the principal diagonal passing through this vertex. Then we have $\epsilon_k(e) = \epsilon_k(e') := \epsilon_k$ for all $k \neq j$ and $\epsilon_j(e) = -\epsilon_j(e')$. Consider the $(n-1)\times(n-1)$ matrix:

$$\bar{Q}_A = \begin{bmatrix} \epsilon_1 \,\text{col}(Q,1) & \dots & \epsilon_{j-1}\,\text{col}(Q,j-1), & \epsilon_{j+1}\,\text{col}(Q,j+1) & \dots & \epsilon_n\,\text{col}(Q,n) \\ 1 & \dots & 1 & 1 & \dots & 1 \end{bmatrix}. \tag{2.13}$$

If its integral Smith form is $\bar{Q}_A^{\text{ismith}} = \text{diag}(t_1 \dots t_{n-1})$, then Y has a singularity of type $\Gamma_A = \mathbb{Z}_{t_1} \times \dots \times \mathbb{Z}_{t_{n-1}}$ at the point Y_A (the common point of $Y_e, Y_{e'}$ and Y_j).[7] The transverse action of this group can be determined as explained in appendix A.

5. If the $\mathbb{Z}_2$ subgroup of the projectivising U(1) acts trivially on X, then the singularity group along Y_j is $\mathbb{Z}_{m_j}$, with transverse action given by (2.12) (with $t_j = m_j$). The singularity groups along the loci Y_e and Y_A are given by the quotients of the groups Γ_e, Γ_A determined at (4) through $\mathbb{Z}_2$ (which is assured to be a subgroup of Γ_e and Γ_A). The action of $\Gamma_e/\mathbb{Z}_2$ and $\Gamma_A/\mathbb{Z}_2$ on the coordinates transverse to Y_e and Y_A is induced by the actions determined at (4) upon taking this quotient.

6. The toric hyperkähler cone X has a T^2 of isometries which preserve its hyperkähler structure. These descend to isometries of the ESD space M and induce isometries of the G_2 cone $\mathcal{C}(Y)$. Such isometries of the G_2 cone will be called *special.*

(6a) Given an edge e of Δ, the associated locus $\mathcal{C}(Y_e)$ in the G_2 cone is (point-wisely) fixed precisely by that U(1) subgroup of the special isometry group whose Lie algebra equals $\mathbb{R}\nu_e$ (viewed as a subalgebra of the Lie algebra $\mathbb{R}^2$ of T^2).

(6b) The locus $\mathcal{C}(Y_j)$ (the cone over Y_j in $\mathcal{C}(Y)$) is (point-wisely) fixed by that U(1) of special isometries whose Lie algebra equals $\mathbb{R}\nu_j$.

Let E_{sing} be the set of edges e of Δ for which Y_e consists of singular points of Y and J_{sing} be the subset of indices $j \in \{1,\dots,n\}$ for which Y_j consists of singular points. A *good isometry* of $\mathcal{C}(Y)$ is a *special isometry* whose fixed point set contains the union $\mathcal{C}(Y_{\text{sing}}) := \left[\cup_{e\in E_{\text{sing}}}\mathcal{C}(Y_e)\right] \cup \left[\cup_{j\in J_{\text{sing}}}\mathcal{C}(Y_j)\right]$. Good isometries form a (possibly trivial) Lie subgroup of the two-torus of special isometries.

[7] This group is expected to be cyclic in our models (and it is cyclic in the models we investigated). However, we shall not attempt to give an independent proof of this statement.

(6c) The Lie algebra of the group of good isometries coincides with the intersection $\left[\cap_{e\in E_{\rm sing}}\mathbb{R}\nu_e\right]\cap\left[\cap_{j\in J_{\rm sing}}\mathbb{R}\nu_j\right]$, when viewed as a subalgebra of the Lie algebra $\mathbb{R}^2$ of T^2. In particular, $\mathcal{C}(Y)$ admits a good isometry if and only if this intersection does not vanish. In this case, the intersection is a one-dimensional space and $\mathcal{C}(Y)$ admits precisely a U(1) group of good isometries.

It is clear from this result that models admitting good isometries are non-generic in our class. If the model admits a good isometry, then we shall argue that its IIA reduction through this isometry can be described in terms of weakly-coupled D6-branes. In this case, the T-dual IIB solution describes a system of delocalized D5-branes. Reduction through isometries which are not good leads to strongly coupled IIA and IIB descriptions.

2.1 Example: a model with one hyperkähler charge

Consider the toric hyperkähler cone $X = \mathbf{H}^3///\,\mathrm{U}(1)$, with the U(1) action given by:

$$Q = [\,1 \quad 2 \quad 2\,]\,. \tag{2.14}$$

This is one of the models studied in [50], whose ESD base is $M = W\mathbb{CP}_{4,3,3}$. The matrix Q has trivial invariant factors, since its integral Smith form is $[\,1 \quad 0 \quad 0\,]$. An integral basis for the kernel of Q is given by the rows of the matrix:

$$G = \ker(Q)^t = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \tag{2.15}$$

whose columns ν_1, ν_2 and ν_3 are the toric hyperkähler generators. Since the last two generators are primitive, the loci $X_2 : u_2 = 0$ and $X_3 : u_3 = 0$ associated with the flats H_2, H_3 are smooth in the hyperkähler cone X. For the non-primitive vector ν_1 we have $m_1 = gcd(\nu_1^1, \nu_1^2) = gcd(2, 0) = 2$, which gives $\mathbb{Z}_2$ singularities along the locus $X_1 : u_1 = 0$. The singularity is worse at the apex $u = 0$. It is clear that the projectivising U(1) acts effectively on X (since the row vector Q contains even entries).

The twistor space $Y = X//_\zeta\,\mathrm{U}(1)$ can be realized as the quadric hypersurface:

$$w_1^{(+)}w_1^{(-)} + 2w_2^{(+)}w_2^{(-)} + 2w_3^{(+)}w_3^{(-)} = 0 \tag{2.16}$$

in the four-dimensional toric variety $\mathbb{T} = (\mathbb{C}^6 - \{0\})/(\mathbb{C}^*)^2 = \mathbb{C}^6//T^2$ defined by the charge matrix:

$$\tilde{Q} = \begin{bmatrix} 1 & 2 & 2 & -1 & -2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \tag{2.17}$$

whose integral Smith form equals:

$$\tilde{Q}^{\rm ismith} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.18}$$

Since $\tilde{Q}$ has trivial invariant factors, the associated T^2 action is effective on $\mathbb{C}^6$.

The distinguished locus is described by the planar polygon:

$$\Delta : |\nu_1 \cdot a| + |\nu_2 \cdot a| + |\nu_3 \cdot a| = \zeta\,, \tag{2.19}$$

where a is a vector in $\mathbb{R}^2$. This is a hexagon whose vertices (numbered $1, \ldots, 6$ in figure 1)

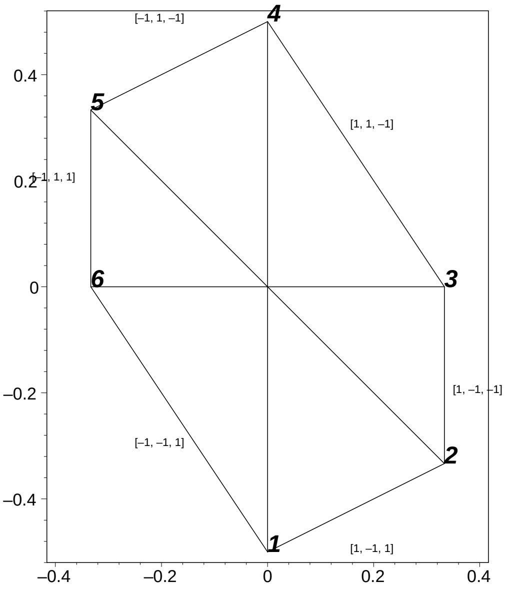

Figure 1: The polygon Δ for $\zeta = 1$.

correspond to the columns of the matrix:

$$\begin{bmatrix} 0 & 1/3 & 1/3 & 0 & -1/3 & -1/3 \\ -1/2 & -1/3 & 0 & 1/2 & 1/3 & 0 \end{bmatrix}. \tag{2.20}$$

Let us consider the sign vectors $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ associated with the various edges as shown in the figure:

$$\begin{aligned} &[12]: \ (+,-,+)\,, \qquad &&[23]: \ (+,-,-)\,, \qquad &&[34]: \ (+,+,-) \\ &[45]: \ (-,+,-)\,, \qquad &&[56]: \ (-,+,+)\,, \qquad &&[61]: \ (-,-,+)\,. \end{aligned} \tag{2.21}$$

These give the vectors $\nu_e := \sum_{j=1}^3 \epsilon_j(e)\nu_j$:

$$\nu_{12} = -\nu_{45} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \nu_{23} = -\nu_{56} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \nu_{34} = -\nu_{61} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{2.22}$$

Since $\nu_{12} = -\nu_{45}$ and $\nu_{34} = -\nu_{61}$ are primitive, the associated edges of Δ correspond to smooth two-spheres in Y. The vectors $\nu_{23} = -\nu_{56}$ give $\mathbb{Z}_3$ singularities along the horizontal spheres associated with the opposite edges [23] and [56]; these are related by the antipodal map of the fibration $Y \to M$.

The locus Y_V consists of two-spheres (fibers of $Y \to M$) associated with the principal diagonals $D_1 = [14]$, $D_2 = [36]$ and $D_3 = [25]$ of Δ, which lie along the lines $h_1 : \nu_1 \cdot a = 0$, $h_2 : \nu_2 \cdot a = 0$ and $h_3 : \nu_3 \cdot a = 0$. To identify the singularity type along these spheres, we compute the integral Smith forms of the matrices:

$$\begin{aligned} \tilde{Q}_1 &= \begin{bmatrix} 2 & 2 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \tilde{Q}_1^{\mathrm{ismith}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \\ \tilde{Q}_2 = \tilde{Q}_3 &= \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \tilde{Q}_2^{\mathrm{ismith}} = \tilde{Q}_3^{\mathrm{ismith}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.23}$$

Thus the S^2 fibers associated with D_2 and D_3 are smooth in Y, while the fiber associated with D_1 gives a sphere's worth of $\mathbb{Z}_4$ singularities.

Conclusion. The twistor space contains two S^2's worth of $\mathbb{Z}_3$ singularities and one S^2 of $\mathbb{Z}_4$ singularities. The $\mathbb{Z}_3$ spheres are horizontal (lifts of spheres lying in the selfdual base M), while the $\mathbb{Z}_4$ sphere is vertical (a fiber of $Y \to M$). The reduced polygon Δ_M is a triangle, which is covered by Δ through the sign inversion in $\mathbb{R}^2$ (figure 2). These conclusions clearly agree with those of [50]. While models of this type are well-understood from the work just cited, the methods of the present paper apply more generally, and lead to many new examples, some of which are discussed in section 8.

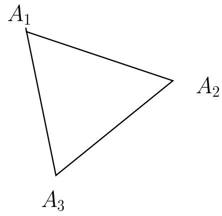

Figure 2: The polygon Δ_M.

3. The basic set-up

3.1 Quaternion-Kähler spaces, twistor spaces and hyperkähler cones

There exists a well-known relation [27] between four types of riemannian geometries, namely quaternion-Kähler, twistor, 3-Sasaki and conical hyperkähler. This connects a $4d$-dimensional hyperkähler cone X with a $4d-4$-dimensional quaternion-Kähler space M, a $4d-2$ -dimensional twistor space Y and a $4d-1$ dimensional 3-Sasaki space S. The various relations are summarized in figure 3, whose arrows are explained below.[8]

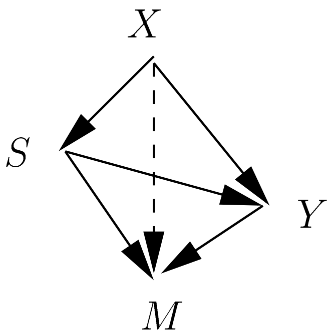

Figure 3: Relation between the four geometries.

We remind the reader that a *hyperkähler cone* is a hyperkähler space X (of real dimension $4d$) which can be written as the metric cone over a compact, $4d-1$ dimensional riemannian space S. Up to a possible finite cover $X \to \tilde{X}$, this amounts [32] to considering a (complete) hyperkähler space $\tilde{X}$ which admits a *hyperkähler potential*, i.e. a real-valued function $\mathcal{K}$ which is a Kähler potential with respect to *all* compatible complex structures. In this case, the vector field Ξ defined through $d\mathcal{K}(.) = -g(\Xi, .)$ (where g is the hyperkähler metric) induces a one-parameter semigroup of homotheties. One can choose the integration constant in $\mathcal{K}$ such that the homothety action rescales it according to $\mathcal{K} \to \alpha^2\mathcal{K}$, where $\alpha > 0$ is the homothety parameter. Accordingly, the hyperkähler potential defines a *radial distance function* $r := \mathcal{K}^{1/2}$, which scales as $r \to \alpha r$ and identifies Ξ with $r\frac{\partial}{\partial r}$. This presents (a cover[9] X of) $\tilde{X}$ as the metric cone over a $4d-1$-dimensional compact space S obtained by restriction to a level $r = \zeta^{1/2} > 0$. Since its metric cone is hyperkähler, S is a *3-Sasaki* space [27]; in particular, the metric induced on S by restriction is Einstein and of positive scalar curvature — the choice of ζ fixes its overall scale. The arrow $X \to S$ in figure 3 stands for this restriction.

[8]In the present paper we only consider 'the positive case'. Thus M will have positive scalar curvature and X will carry a positive-definite metric. The 'negative case' is relevant for applications to hypermultiplet moduli spaces in supergravity, and part of our considerations can be extended to that situation.

[9]Taking a finite cover of X may be required in order to insure compactness of S.

The hyperkähler cone admits an isometric (but not triholomorphic) $\mathrm{Sp}(1)$ action which preserves the hyperkähler potential and thus the distance function r; its generators are the vector fields $I\Xi = I(r\partial_r)$, where I are the compatible complex structures of X. Existence of the hyperkähler potential implies [32] that this action rotates (i.e. acts transitively on) the compatible complex structures of X. Upon fixing some I, one obtains a group isomorphism $\mathrm{Sp}(1) \approx \mathrm{SU}(2)$ which is dependent of this choice; the totality of such isomorphisms is parameterized by a two-sphere. Then the diagonal $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(2)$ acts on X with generator $I\Xi$; this action preserves the Kähler form of I. The hyperkähler potential $\mathcal{K} = r^2$ coincides with the Kähler moment map of this $\mathrm{U}(1)$ action. In fact, the $\mathrm{U}(1)$ may have a trivially acting $\mathbb{Z}_2$ subgroup (the diagonal subgroup $\{1, -1\}$ of $\mathrm{Sp}(1)$), so the effectively acting group $\mathrm{U}(1)_{\mathrm{eff}}$ is either $\mathrm{U}(1)$ or $\mathrm{U}(1)/\mathbb{Z}_2$. The quotient $Y := S/\mathrm{U}(1)_{\mathrm{eff}} = \mathcal{K}^{-1}(\zeta)/\mathrm{U}(1)_{\mathrm{eff}}$ is the Kähler reduction of X at level ζ. The map $X \to Y$ in figure (3) stands for this quotient, while the map $S \to Y$ stands for the associated $\mathrm{U}(1)_{\mathrm{eff}}$ quotient of S. Using the fact that S is 3-Sasaki (or that the cone X is hyperkähler), one shows [27] that Y is a *twistor space* [17, 18, 51, 35], i.e. a projective complex Fano[10] variety admitting a so-called *complex contact structure*[11] [17, 18, 51] and a Kähler-Einstein metric of positive curvature. The Kähler-Einstein metric on Y is induced from the hyperkähler metric of X by the Kähler reduction. Such a variety can always be written as the twistor space $tw(M)$ (in the sense of [17, 18]) of a quaternion-Kähler space M, which is uniquely determined by this property [51]. For this, one uses the complex contact structure in order to build a fibration of Y through holomorphically embedded 2-spheres, whose normal bundle in Y has the form $O_{\mathbb{P}^1}(1)^{\oplus d}$. The space M is recovered as the base of this fibration, i.e. the image of the map p which contracts the $\mathbb{P}^1$ fibers. Then the contact distribution of Y coincides with the horizontal distribution of this fibration, and the Kähler-Einstein metric of the twistor space induces a quaternion-Kähler metric on M with respect to which p becomes locally a riemannian submersion. The arrow $Y \to M$ in figure (3) describes this process of passing from Y to M.

The arrow $S \to M$ can be described as follows. As mentioned above, the hyperkähler cone X admits an isometric $\mathrm{Sp}(1)$ action which preserves the distance function r. This action restricts to the 3-Sasaki space S. It turns out that the diagonal $\mathbb{Z}_2$ subgroup $\{1, -1\}$ of $\mathrm{Sp}(1)$ may act trivially, which means that the effectively acting group $\mathrm{Sp}(1)_{\mathrm{eff}}$ is either $\mathrm{Sp}(1)$ or $\mathrm{Sp}(1)/\mathbb{Z}_2 = \mathrm{SO}(3)$. The quotient of S through this action coincides metrically and topologically with the quaternion-Kähler space M. In particular, the $\mathrm{Sp}(1) = S^3$ or $\mathrm{SO}(3)$ fibration map $S \to M$ is locally a riemannian submersion.

[10] We remind the reader that a complex variety Y is *Fano* if $c_1(TY)$ is positive, i.e. the anticanonical line bundle K_Y^{-1} is ample.

[11] A *complex contact structure* [17, 18, 51] on a complex variety Y is a maximally non-integrable holomorphic Frobenius distribution on Y, i.e. a corank one holomorphic subbundle D of the holomorphic tangent bundle TY, with the property that the Frobenius obstruction map $D \times D \xrightarrow{[\cdot,\cdot]} TY/D$ is nondegenerate everywhere. In this case, the holomorphic line bundle $L := TY/D$ is called the *contact line bundle* of D. If Y has complex dimension $2d - 1$, then it easy to see [51] that L^d is isomorphic with the anticanonical line bundle of Y, so that L is a d^{th} root of K_Y^{-1}.

Since this procedure leads to the same space M, the full arrows in figure (3) commute. The dashed vertical arrow $X \to M$ is defined as the composition of the two arrows on the left, which equals the composition of the two arrows on the right. This describes the so-called *conformal quotient* of [48, 31, 49], which presents M as $\mathcal{K}^{-1}(\zeta)/\operatorname{Sp}(1)_{\text{eff}} = S/\operatorname{Sp}(1)_{\text{eff}}$.

The correspondences shown in figure 3 admit certain inverses, which can be described as follows. Given the 3-Sasaki space S, one recovers X as the metric cone over S. Given the twistor space Y, one obtains X by a procedure due to LeBrun [17, 51, 32], which can be described as follows. If K_Y is the canonical line bundle of Y, then one considers the *contact line bundle* $L := K_Y^{-1/d}$ (the precise root is determined by the contact structure of Y) and the space $(L^{-1})^\times = (K_Y^{1/d})^\times$ obtained from the total space of the dual line bundle L^{-1} by removing its zero section. LeBrun shows that this carries a hyperkähler structure, which covers the well-known Ricci-flat Kähler structure on $K_Y^\times$ due to Calabi. The hyperkähler cone X is obtained from $(L^{-1})^\times$ by completing this hyperkähler metric, which topologically has the effect of replacing the zero section of L^{-1} with the apex of X (thus identifying all points of this zero section to a single point). The 3-Sasaki space S is recovered as the sphere bundle associated with L^{-1}, taken with respect to the metric induced on L^{-1} by the Kähler-Einstein metric of Y. If $||.||_{L^{-1}}$ is the fiberwise norm with respect to this metric, then the radial distance function of X is $r = ||.||_{L^{-1}}$, and S is recovered by imposing the condition $r^2 = \zeta$ in each fiber. The homothety action on X is given by the standard fiberwise rescaling $u \to \alpha u$, while the U(1) action on X associated with the complex structure induced by L^{-1} is given by the fiberwise action $u \to e^{2\pi i\phi}u$. The 3-Sasaki space and hyperkähler cone constructed in this manner carry an effective action of SO(3) (so that $\operatorname{Sp}(1)_{\text{eff}} = \mathrm{SO}(3)$, $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)/\mathbb{Z}_2$ and $Y = S/\mathrm{U}(1)_{\text{eff}} = S/(\mathrm{U}(1)/\mathbb{Z}_2)$). Whether this lifts to an effective Sp(1) action on some other 3-Sasaki space S' or hyperkähler cone X' depends on whether the contact line bundle admits a square root L'. In this case, one can obtain S' and X' by repeating the construction with L replaced by L'.

Given the quaternion-Kähler space M, its twistor space is recovered through the standard construction of Salamon and Bergery [17, 18], while the 3-Sasaki space S can be extracted as explained in [1, 27]. The latter construction recovers S as the principal SO(3) bundle associated with the subbundle $\mathcal{G} \subset \operatorname{End}(TM)$ which specifies the quaternion-Kähler structure of M. Finally, the hyperkähler cone X can be recovered from M through the construction of [32]. This constructs a hyperkähler metric on a principal $\mathbf{H}^*/\mathbb{Z}_2$ bundle (the Swann bundle) over M, which can be completed to a metric on the hyperkähler cone X. The total space of the Swann bundle $\mathcal{U}(M)$ coincides with the space $(L^{-1})^\times$ in LeBrun's construction, and the hyperkähler metric on $\mathcal{U}(M)$ constructed in [32] agrees with the hyperkähler metric on $(L^{-1})^\times$. Whether the SO(3) bundle S lifts to an Sp(1) bundle (equivalently, whether Swann's bundle lifts to an $\mathbf{H}^*$-bundle) depends on the topology of M and is decided by vanishing of the so-called Marchiafava-Romani class [19, 17]. This is essentially the same obstruction as the existence of a square root of the contact line bundle L mentioned above. If this obstruction vanishes, then one has *two* 3-Sasaki spaces

S, S' and *two* hyperkähler cones X, X' associated with M, even though the twistor space Y is uniquely determined. The two choices differ in whether the $\mathbb{Z}_2$ subgroup of $\mathrm{Sp}(1)$ acts trivially (S, X) or nontrivially (S', X'), with S' being a double cover of S and X' a double cover of X (figure 4). This $\mathbb{Z}_2$ ambiguity will be important in section 6.

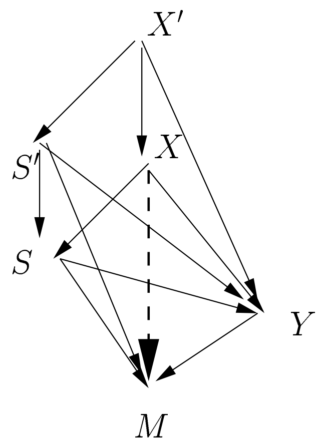

Figure 4: Diagram of 'inverse' geometries when M has vanishing Marchiafava-Romani class.

3.2 G_2 cones from quaternion-Kähler spaces

3.2.1 The construction

In the case $d = 2$ (so that M is a 4-dimensional quaternion-Kähler space), a construction due to [37] and [38] allows one to obtain a metric of G_2 holonomy from the metric on a cone $\mathcal{C}(Y)$ built over the twistor space $Y = tw(M)$. The metric of [37, 38] is obtained as follows. First, let us recall that in the four-dimensional (positive) case, a four-manifold M is quaternion-Kähler if it admits a self-dual metric which is Einstein and of positive scalar curvature. In this situation, the twistor space of M can be (topologically) identified with the sphere bundle associated with the bundle $\Lambda^{2,-}(T^*M)$ of antiselfdual two-forms on M:

$$Y = S(\Lambda^{2,-}(T^*M))\,. \tag{3.1}$$

Since the anti-selfduality condition is invariant under conformal transformations of the metric on M, it follows that Y depends only on the conformal equivalence class of this metric. This is a special property of twistor spaces associated with quaternion-Kähler four-manifolds. If $d\sigma^2$ is the self-dual Einstein metric on M, then the Kähler-Einstein metric on the twistor space has the form:

$$d\rho^2 = |d\sigma|^2 + |d_A u|^2\,, \tag{3.2}$$

where $u = (u_1, u_2, u_3)$ is a local frame of sections of $\Lambda^{2,-}(T^*M)$ with $\sum_{i=1}^3 u_i^2 = 1$ and A is the connection induced on this bundle by the Levi-Civita connection of M.

The construction of [37, 38] proceeds as follows. First, one considers the following *modified* metric on Y:

$$d\rho'^2 = \frac{1}{2}\left[d\sigma^2 + \frac{1}{2}|d_A u|^2\right]. \tag{3.3}$$

The G_2 cone $\mathcal{C}(Y)$ of [37, 38] is simply the metric cone built over the riemannian space $(Y, d\rho'^2)$:

$$ds^2 = dr^2 + r^2 d\rho'^2 = dr^2 + \frac{r^2}{2}\left(d\sigma^2 + \frac{1}{2}|d_A u|^2\right). \tag{3.4}$$

The topological space obtained by removing the apex of $\mathcal{C}(Y)$ can be identified with $(\Lambda^{2,-}(T^*Y))^\times$, the space obtained from the total space of $\Lambda^{2,-}(T^*Y)$ by removing its zero section. In this case, the coordinate r is identified with the radial coordinate inside each fiber. Then $\mathcal{C}(Y)$ is obtained by completing the metric (3.4) on $(\Lambda^{2,-}(T^*Y))^\times$. The metric (3.4) admits a one-parameter family of deformations of the form:

$$ds^2 = \frac{1}{1-(r_0/r)^4}dr^2 + \frac{r^2}{2}\left(d\sigma^2 + \frac{1}{2}\left(1-\left(\frac{r_0}{r}\right)^4\right)|d_A u|^2\right), \qquad r_0 \geq 0 \qquad (3.5)$$

whose members have G_2 holonomy; the conical limit is obtained for $r_0 = 0$. The conical singularity is smoothed out in the metrics (3.5), though other singularities are still present. These partially resolved G_2 metrics are complete on the bundle $\Lambda^{2,-}(T^*M)$ (with its zero section included).

3.2.2 Special isometries

It is clear from (3.2), (3.3) that any isometry of the Einstein self-dual metric on M will lift to an isometry of both the Kähler-Einstein metric ρ and the modified metric ρ' on $Y = tw(M)$. Through the explicit constructions (3.4) and (3.5), such a symmetry of M also lifts to an isometry of the conical and deformed G_2 metrics. This observation will be important in later sections, when we will study isometries of M which are induced from its hyperkähler cone X; according to the discussion above, such symmetries will automatically translate into isometries of the associated G_2 metric, which can therefore be used to extract the type-II interpretation of our models.

3.3 From hyperkähler cones to G_2 cones

As recalled above, the study of quaternion-Kähler spaces can be reduced to that of hyperkähler cones. In particular, if one starts with an 8-dimensional hyperkähler cone, one has an associated Einstein self-dual space, and therefore an associated G_2 cone. This allows us to produce (and study) large families of G_2 cones by starting with eight-dimensional hyperkähler cones. Since we are interested in understanding M-theory physics on the former, we shall pay particular attention to their singularities. It is clear from the explicit construction of (3.4) that the singularities of the G_2 cone are completely determined from the singularities of the twistor space of M, and that the nature of the former does not depend on which of the metrics ρ and ρ' one uses on Y. This allows us to determine the singularities of $\mathcal{C}(Y)$ by studying the singularities of the twistor space Y, without worrying about the metric change $\rho \to \rho'$ involved in the construction (3.4). The singularities of Y will be determined by studying its presentation $Y = X//_{>0}\mathrm{U}(1)$ as a Kähler quotient of the hyperkähler cone X. With arbitrary hyperkähler cones, the study of singularities is rather involved. To simplify the problem, we shall limit ourselves to a particular class of cones X, namely those which are *toric hyperkähler*. This will allow us to study the singularities of Y in a systematic manner, and therefore extract the singularities of the associated G_2 cone, by applying methods reminiscent of those familiar from toric geometry [42, 44, 45, 43, 47, 46].

4. Toric hyperkähler spaces

Toric hyperkähler spaces form a hyperkähler analogue of the well-known toric varieties which play a central role in many subjects of algebraic geometry. To define them, we start with a brief description of torus actions on affine quaternion spaces.

4.1 Torus actions on affine quaternion spaces

Let $\mathbf{H}$ denote the field of quaternions, with quaternion units $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. The field $\mathbb{C}$ of complex numbers embeds into $\mathbf{H}$ upon identifying the complex units $1, i$ with the quaternion units $1, \mathbf{i}$. This is the only embedding of $\mathbb{C}$ into $\mathbf{H}$ which will be used in this paper. Given a quaternion $u = u^0 + \mathbf{i}u^1 + \mathbf{j}u^2 + \mathbf{k}u^3$ (with $u^0, \ldots, u^3$ some real numbers), one defines its conjugate by $\overline{u} = u^0 - \mathbf{i}u^1 - \mathbf{j}u^2 - \mathbf{k}u^3$ and its norm by $||u||^2 = \overline{u}u = (u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2$.

Consider the quaternion affine space $\mathbf{H}^n$, with quaternion coordinates $u_1, \ldots, u_n \in \mathbf{H}$. This carries the standard flat metric:

$$ds^2 = \sum_{k=1}^{n} d\overline{u}_k du_k \,, \tag{4.1}$$

which induces the norm $||u||^2 = \sum_{k=1}^{n} ||u_k||^2$. When endowed with this metric, $\mathbf{H}^n$ becomes a hyperkähler manifold, whose three compatible complex structures are induced by component-wise multiplication from the *right* with the quaternion units $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{I}(u) = (u_1\mathbf{i} \ldots u_n\mathbf{i}) \,, \qquad \mathbf{J}(u) = (u_1\mathbf{j} \ldots u_n\mathbf{j}) \,, \qquad \mathbf{K}(u) = (u_1\mathbf{k} \ldots u_n\mathbf{k}) \tag{4.2}$$

4.1.1 The diagonal T^n action

The space $\mathbf{H}^n$ admits an obvious action by the n-dimensional torus $T^n = \mathrm{U}(1)^n$:

$$u = (u_1, \ldots, u_n) \to u' = (\Lambda_1 u_1, \ldots, \Lambda_n u_n) \,, \tag{4.3}$$

where Λ_k are complex numbers of unit modulus. In (4.3), we view Λ_k as elements of $\mathbf{H}$, and use juxtaposition to denote quaternion multiplication. This action preserves the metric (4.1) and commutes with the three complex structures (4.2). Therefore, it admits a hyperkähler moment map $\eta : \mathbf{H}^n \to (\mathrm{Im}\,\mathbf{H})^n$ with components:

$$\eta^{(j)}(u) = \overline{u}_j \mathbf{i} u_j \in \mathrm{Im}\,\mathbf{H} \,, \qquad j = 1, \ldots, n \,. \tag{4.4}$$

We remind the reader that the space $\mathrm{Im}\,\mathbf{H}$ of imaginary quaternions is the real vector subspace of $\mathbf{H}$ spanned by the imaginary quaternion units $\mathbf{i}, \mathbf{j}, \mathbf{k}$. This 3-dimensional space can be identified with $\mathbb{R}^3$ (i.e. $\mathrm{Im}\,\mathbf{H} = \langle\, \mathbf{i},\ \mathbf{j},\ \mathbf{k}\, \rangle_{\mathbb{R}} \approx \mathbb{R}^3 = \vec{\mathbb{R}}$) through the map which sends $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the canonical basis e_1, e_2, e_3 of $\mathbb{R}^3$. Upon writing $\eta^{(j)} = \mathbf{i}\eta_1^{(j)} + \mathbf{j}\eta_2^{(j)} + \mathbf{k}\eta_3^{(j)}$, one obtains a map $\vec{\eta}^{(j)} = (\eta_1^{(j)}, \eta_2^{(j)}, \eta_3^{(j)}) : \mathbf{H}^n \to \mathbb{R}^3$ and a map $\vec{\eta} := (\vec{\eta}^{(1)} \ldots \vec{\eta}^{(n)}) : \mathbf{H}^n \to (\mathbb{R}^3)^n = \mathbb{R}^{3n}$. It will be convenient to use the following notation. We let $\vec{\mathbb{R}} := \mathbb{R}^3$, and $\vec{\mathbb{R}}^n = (\mathbb{R}^3)^n \approx \mathbb{R}^{3n}$. The arrow superscript indicates that we use the particular presentation of $\mathbb{R}^{3n}$ as a direct product of n copies of $\vec{\mathbb{R}}$, and amounts to thinking of $\mathbb{R}^{3n}$ as the tensor product $\mathbb{R}^3 \otimes \mathbb{R}^n$. We also let $\vec{\eta}_s = (\eta_s^{(1)} \ldots \eta_s^{(n)}) \in \mathbb{R}^n$, for $s = 1, 2, 3$. Similar notations and conventions will be used for all hyperkähler moment maps appearing in this paper.

4.1.2 Description of subtori of T^n through maps of lattices

Any *injective* integral linear map $q^* : \mathbb{Z}^r \to \mathbb{Z}^n$ induces a map of tori $T^r \to T^n$, by tensoring from the right with the abelian group $\mathrm{U}(1)$ (indeed, one has $T^m = \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathrm{U}(1)$

for all m). When combined with (4.3), this gives a T^r action on $\mathbf{H}^n$. If the injective map q^* is otherwise arbitrary, then the induced map of tori need not be injective (in technical language, tensoring with U(1) is not a left exact operation). In this case, T^r can not be regarded as a sub-torus of T^n, and the induced T^r action on $\mathbf{H}^n$ will fail to be effective. To understand when this occurs, let us consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{q^*} \mathbb{Z}^n \xrightarrow{g} A \longrightarrow 0 \tag{4.5}$$

obtained by computing the cokernel of q^* (thus $A = \mathbb{Z}^n/q^*(\mathbb{Z}^r)$). As explained in appendix A, this induces an exact sequence of the form:

$$0 \longrightarrow \Gamma \longrightarrow T^r \longrightarrow T^n \longrightarrow A \otimes_{\mathbb{Z}} \mathrm{U}(1) \longrightarrow 0\,, \tag{4.6}$$

where the group Γ coincides with the torsion subgroup of A. Therefore, the map $T^r \to T^n$ will be an embedding if and only if A contains no torsion; in this case, A coincides with the lattice $\mathbb{Z}^d$, where $d = n - r$. Moreover, this happens (See appendix A) if and only if the charge matrix Q associated with the transpose map $q : \mathbb{Z}^n \to \mathbb{Z}^r$ has trivial 'torsion coefficients', which means that its integral Smith form equals:

$$Q^{\text{ismith}} = [\, I_r \quad 0_{r\times d}\,]\,, \tag{4.7}$$

where I_r denotes the $r \times r$ identity matrix. Equivalently, the r^{th} *discriminantal divisor*[12] of Q equals one.

Throughout this paper, we shall assume that this condition is satisfied. Then (4.5) reduces to a short exact sequence of lattices:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{q^*} \mathbb{Z}^n \xrightarrow{g} \mathbb{Z}^d \longrightarrow 0\,, \tag{4.8}$$

and (4.6) collapses to a short exact sequence of tori:

$$0 \longrightarrow T^r \longrightarrow T^n \longrightarrow T^d \longrightarrow 0\,. \tag{4.9}$$

The $d \times n$ matrix G of the cokernel map g will be called 'the matrix of generators'. Its columns ν_j are integral vectors belonging to the lattice $\mathbb{Z}^d$ and will be called 'toric hyperkähler generators'. We note that the *rows* of G are primitive and form a basis for the kernel of the matrix Q.

4.2 Subtorus actions

With these assumptions, the induced T^r action on $\mathbf{H}^n$:

$$u_j \to u'_j = \prod_{\alpha=1}^{r} \lambda_\alpha^{q_j^{(\alpha)}} u_j\,, \tag{4.10}$$

[12] The r^{th} discriminantal divisor $\mathfrak{g}$ of Q is defined as the greatest common divisor of all of its $r \times r$ minor determinants. One has $t_1, \ldots, t_r = \mathfrak{g}$ (appendix A).

is effective. In this relation, $q_j^{(\alpha)}$ stands for the entry $Q_{\alpha j}$ of the matrix Q. The map q^* describes the embedding:

$$\Lambda_j = \prod_{\alpha=1}^{r} \lambda_\alpha^{q_j^{(\alpha)}} \tag{4.11}$$

of T^r into T^n.

It is obvious that the T^r action (4.10) is tri-holomorphic on $\mathbf{H}^n$ (since so is the T^n action (4.3)). Accordingly, it has a hyperkähler moment map $\mu : \mathbf{H}^n \to \vec{\mathbb{R}}^r$ whose components are given by:

$$\mu^{(\alpha)} := \sum_{j=1}^{n} q_j^{(\alpha)} \overline{u}_j \mathbf{i} u_j \in \operatorname{Im} \mathbf{H}\,. \tag{4.12}$$

As in subsection 4.1.1, we let $\vec{\mu} : \mathbf{H}^n \to \vec{\mathbb{R}}^r$ be the map resulting from the identification $\operatorname{Im} \mathbf{H} = \mathbb{R}^3 = \vec{\mathbb{R}}$. If we let $\vec{q}$ be the map from $\mathbb{R}^n$ to $\vec{\mathbb{R}}^r$ induced by q upon tensoring with $\vec{\mathbb{R}} = \mathbb{R}^3$, then we have $\vec{\mu} = \vec{q} \circ \vec{\eta}$.

4.3 Toric hyperkähler spaces

With the notations of the previous subsection, a *toric hyperkähler space* is defined [34] as the hyperkähler reduction of $\mathbf{H}^n$ through a subtorus T^r of T^n at some level $\vec{\xi} \in \vec{\mathbb{R}}^r$ of the associated hyperkähler moment map:

$$\mathbf{X} = \mathbf{H}^n /// _{\vec{\xi}} T^r = \vec{\mu}^{-1}(\vec{\xi})/T^r\,. \tag{4.13}$$

This description immediately shows that $\mathbf{X}$ is endowed with a hyperkähler structure induced via the reduction process [39]. In general, the space $\mathbf{X}$ will be singular.

In spite of the formal similarity of their definition, the geometry of toric hyperkähler spaces differs qualitatively from that of toric varieties (which result upon performing *Kähler* torus quotients of some *complex* affine space $\mathbb{C}^n$). For example, it can be shown [34] that the topology of X is independent of the choice of hyperkähler moment map levels $\vec{\xi}$, as long as the latter lie in the complement of a codimension *three* subset of $\mathbb{R}^{3r}$ (i.e. the resulting spaces for different values of $\vec{\xi}$ are homeomorphic). This is in marked contrast with the behavior of toric varieties, for which topology changing transitions occur on walls (i.e. subspaces of codimension *one*) in the space of Kähler moment map levels.

Observation 1. In this paper, we do *not* assume that the toric hyperkähler generators ν_j are primitive. In fact, these vectors fail to be primitive even for the simplest examples one wishes to consider, namely the models based on the construction of [36], which were studied in [50] by methods different from ours. This means that our hyperkähler spaces are *more general* than those studied in the work of [34], which assumes primitivity of generators for the most part. Due to this, we will have to modify and adapt some basic results of [34]. Some of the facts we shall use require a direct proof, which can be found in the appendices.

Observation 2. We warn the reader that the term 'toric hyperkähler spaces' is used somewhat ambiguously in the literature. In [20, 22, 23], this language is used to describe quotients of $\mathbf{H}^n$ by abelian groups which are *not* subtori of T^n; the resulting spaces are 'toric' inasmuch as they admit a triholomorphic torus action, but they do not satisfy our definition. The prototypical example of this type is the euclidean Taub-Nut space, which can be obtained upon dividing $\mathbf{H}^2$ through the non-compact group $(\mathbb{R},+)$, acting through transrotations (see [23]). In particular, this construction allows the authors of [20, 23] to give a hyperkähler quotient description of the generalizations of Taub-Nut metrics studied in [21]. We stress that the spaces considered in [20, 22, 23, 21] are *not* toric hyperkähler according to our definition (which agrees with that of [34]).

5. Toric hyperkähler cones

Since we are interested in hyperkähler cones, we shall henceforth concentrate on the case of *vanishing* moment map level $\vec{\xi}=0$ in definition (4.13). Then X admits a one-parameter semigroup of homotheties induced by the obvious rescaling of coordinates of $\mathbf{H}^n$:

$$u_j \to u_j \alpha\,, \qquad \alpha > 0\,. \tag{5.1}$$

It is clear that these transformations descend to homotheties of $X = \mathbf{H}^n///_0 T^r$ in such a way that X becomes a hyperkähler cone. We note that such cones have been studied in [48] from a local perspective.

5.1 Description as a Kähler quotient and embedding in a toric variety

5.1.1 Complex coordinates and Kähler quotient description

Toric hyperkähler cones can be presented as Kähler quotients of affine algebraic varieties. For this, let us write $u_j = u_j^0 + \mathbf{i}u_j^1 + \mathbf{j}u_j^2 + \mathbf{k}u_j^3$ and introduce complex coordinates through $w_j^{(+)} = u_j^0 + \mathbf{i}u_j^1$ and $w_j^{(-)} = u_j^2 - \mathbf{i}u_j^3$. Then:

$$u_j = w_j^{(+)} + \mathbf{j}w_j^{(-)}\,. \tag{5.2}$$

This amounts to endowing $\mathbf{H}^n$ with the complex structure $\mathbf{I}$. The flat hyperkähler metric (4.1) becomes the standard Kähler metric on $\mathbb{C}^{2n}$:

$$ds^2 = \sum_{j=1}^{n}\left(d\overline{w}_j^{(+)}dw_j^{(-)} + d\overline{w}_j^{(-)}dw_j^{(+)}\right), \tag{5.3}$$

while the triholomorphic T^r action (4.10) takes the form:

$$w_j^{(+)} \to \prod_{\alpha=1}^{r}\lambda_\alpha^{q_j^{(\alpha)}}\, w_j^{(+)}\,, \qquad w_j^{(-)} \to \prod_{\alpha=1}^{r}\lambda_\alpha^{-q_j^{(\alpha)}}\, w_j^{(-)}\,. \tag{5.4}$$

The minus sign in the second exponent follows from the anticommutation relation $\mathbf{ij}=-\mathbf{ji}$, which implies $\lambda_\alpha\mathbf{j} = \mathbf{j}\overline{\lambda}_\alpha = \mathbf{j}\lambda_\alpha^{-1}$. This action preserves the Kähler structure of $\mathbb{C}^{2n}$. The hyperkähler moment map separates as $\vec{\mu} = 2[\mathbf{i}\mu_r + \mathbf{k}\mu_c]$, with:

$$\begin{aligned}\mu_r^{(\alpha)}(w) &= \frac{1}{2}\mu_1^{(\alpha)}(w) = \frac{1}{2}\sum_{j=1}^{n} q_j^{(\alpha)}\left(|w_j^{(+)}|^2 - |w_j^{(-)}|^2\right) \in \mathbb{R}\\ \mu_c^{(\alpha)}(w) &= \frac{1}{2}[\mu_3^{(\alpha)}(w) + i\mu_2^{(\alpha)}(w)] = \sum_{j=1}^{n} q_j^{(\alpha)} w_j^{(+)} w_j^{(-)} \in \mathbb{C}\,.\end{aligned} \tag{5.5}$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "q^*"] \arrow[d, "id"'] & \mathbb{Z}^n \arrow[r, "g"] \arrow[d, "j"'] & \mathbb{Z}^d \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "\hat{q}^*"] & \mathbb{Z}^{2n} \arrow[r, "\hat{g}"] & \mathbb{Z}^{2d+r} \arrow[r] & 0
\end{tikzcd}
$$

Figure 5: Embedding a hyperkähler cone in a toric variety.

The levels decompose accordingly as $\vec{\xi} = (\xi_r, \xi_c) \in \mathbb{R}^r \oplus \mathbb{C}^r \approx \mathbb{R}^{3r}$. The real component μ_r is the Kähler moment map for the action (5.4) on $\mathbb{C}^{2n}$.

These decompositions allow us to view the hyperkähler cone X as a Kähler quotient:

$$X = \mathcal{Z}//_0 T^r\,, \tag{5.6}$$

where $\mathcal{Z}$ is the affine algebraic variety defined through:

$$\mathcal{Z} = \mu_c^{-1}(0) \subset \mathbb{C}^{2n}\,. \tag{5.7}$$

The presentation (5.6) also gives X as the solution set of the quadric equations:

$$\sum_{k=1}^{n} q_k^{(\alpha)} w_k^{(+)} w_k^{(-)} = 0 \qquad (\alpha = 1, \ldots, r) \tag{5.8}$$

in the toric variety $\mathbb{S} = \mathbb{C}^{2n}//_0 T^r \approx \mathbb{C}^{2n}/(\mathbb{C}^*)^r$. This shows that hyperkähler cones are algebraic varieties of a familiar type.

5.1.2 Toric description of the ambient space

It is straightforward to extract the toric description of $\mathbb{S}$, which we give below for the sake of completeness. First notice that the T^r action (5.4) on the complex coordinates has the $r \times (2n)$ charge matrix:

$$\hat{Q} = [\,Q, -Q\,]\,, \tag{5.9}$$

which defines the lattice map $\hat{q}^* = (q^*, -q^*) : \mathbb{Z}^r \to \mathbb{Z}^{2n}$. It is shown in appendix E that the cokernel of $\hat{q}^*$ has no torsion, so one obtains an exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\hat{q}^*} \mathbb{Z}^{2n} \xrightarrow{\hat{g}} \mathbb{Z}^{2d+r} \longrightarrow 0\,. \tag{5.10}$$

This completely specifies the ambient toric variety $\mathbb{S}$. The columns of the $(2d+r) \times (2n)$ matrix $\hat{G}$ (the matrix of the map $\hat{g}$ with respect to the canonical bases of $\mathbb{Z}^{2n}$ and $\mathbb{Z}^{2d+r}$) are the toric generators of $\mathbb{S}$.

Considering (4.8) and (5.10), the 3-lemma shows that there exists a uniquely-determined and injective map $f : \mathbb{Z}^d \to \mathbb{Z}^{2d+r}$ which makes the diagram of figure 5 commute.

Thus there exists a unique $(2d+r) \times d$ matrix F satisfying the constraint:

$$FG = \hat{G}J\,, \tag{5.11}$$

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$$
\begin{tikzcd}
X \arrow[r] \arrow[dr, "\vec{\pi}"'] & \mathcal{N} \arrow[r] \arrow[d, "\vec{\pi}_0"'] & \mathbf{H}^n \arrow[d, "\vec{\eta}"'] \arrow[dr, "\vec{\mu}"] & & \\
0 \arrow[r] & \vec{\mathbb{R}}^d \arrow[r, "\vec{g}^*"'] & \vec{\mathbb{R}}^n \arrow[r, "\vec{q}"'] & \vec{\mathbb{R}}^r \arrow[r] & 0
\end{tikzcd}
$$

Figure 6: Construction of the moment map for the induced T^d action on X.

where $J = (I_n, -I_n)$ is the $(2n) \times n$ matrix of the map $j = (id, -id) : \mathbb{Z}^n \to \mathbb{Z}^{2n}$. Since f is injective, the matrix F has maximal rank. Its columns span a d-dimensional subspace of $\mathbb{R}^{2d+r}$, which can be identified with the Lie algebra of the triholomorphic T^d action on X. In particular, f embeds the toric hyperkähler generators ν_j as integral vectors $f(\nu_j)$ lying in this subspace.

5.2 Description as a T^d fibration

It is clear from the hyperkähler quotient construction that the torus $T^d = T^n/T^r$ will act on X preserving its hyperkähler structure. In particular, one has an associated hyperkähler moment map $\vec{\pi} : X \to \mathbb{R}^{3d}$. It is shown in [34] that this map is surjective and descends to a homeomorphism between X/T^d and $\mathbb{R}^{3d}$. In particular, X is connected and can be viewed as a T^d fibration over the *entire*[13] space $\mathbb{R}^{3d}$.

5.2.1 Construction of the fibration

To see this explicitly, we notice that the level set $\mathcal{N} := \vec{\mu}^{-1}(0) \subset \mathbf{H}^n$ can be described as follows. Since $\vec{\mu} = \vec{q} \circ \vec{\eta}$, a point $u \in \mathbf{H}^n$ belongs to $\mathcal{N}$ if and only if $\vec{\eta}(u)$ belongs to the kernel of $\vec{q}$. By dualizing (4.8), we obtain a short exact sequence:

$$
0 \longrightarrow \mathbb{Z}^d \xrightarrow{g^*} \mathbb{Z}^n \xrightarrow{q} \mathbb{Z}^r \longrightarrow 0\,, \tag{5.12}
$$

which shows that $\ker q = img^*$. Since g^* is injective, we find that there exists a unique map $\vec{\pi}_0 : \mathcal{N} \to \vec{\mathbb{R}}^d$ such that $\vec{\eta}|_{\mathcal{N}} = \vec{g}^* \circ \vec{\pi}$. This map presents $\mathcal{N}$ as a T^n fibration over $\vec{\mathbb{R}}^d$. Moreover, it descends to a well-defined map $\vec{\pi} : X = \mathcal{N}/T^r \to \vec{\mathbb{R}}^d$, since $\mathcal{N}$ and $\vec{\mu}$ are T^r-invariant; this amounts to writing X as a T^d fibration over $\mathbb{R}^{3d}$, obtained from $\mathcal{N}$ by quotienting out its T^n fibers through the subtorus $T^r \subset T^n$. It is easy to see that $\vec{\pi}$ is surjective and coincides with the hyperkähler moment map of the induced T^d action on X. This argument is summarized in figure 6.

In words, a point $u = (w^{(+)}, w^{(-)}) \in \mathbf{H}^n$ satisfies the moment map equations $\vec{\mu}(u) = 0$ (and thus belongs to $\mathcal{N}$) if and only if there exists a vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ such that $\vec{\eta}(u) = \vec{g}^*(\vec{v})$, i.e.:

$$
\eta_r(u) = \frac{1}{2}(|w_j^{(+)}|^2 - |w_j^{(-)}|^2) = \nu_j \cdot a\,, \qquad \eta_c(u) = w_j^{(+)} w_j^{(-)} = \nu_j \cdot b\,, \quad \text{for all } j = 1, \ldots, n\,, \tag{5.13}
$$

[13]This should be contrasted with the situation for toric varieties. In that case, the image of the Kähler moment map of the densely embedded torus is only a convex subset of real affine space, namely the Delzant polytope of the variety.

where $a = \frac{v_1}{2}$, $b = \frac{v_3+iv_2}{2}$ and $\cdot$ stands for the standard scalar product. In this case, we have $\vec{\pi}_0(u) = \vec{v}$ and similarly for the induced map π, so that:

$$\pi_r(u) = a \qquad \text{and} \qquad \pi_c(u) = b\,. \tag{5.14}$$

Passage to the 'dual variables' $\vec{v}$ allows one to 'solve' the moment map constraints $\vec{\mu} = 0$. This is a well-known observation familiar from the work of [39, 48], which we have simply reformulated in 'invariant' language. One can describe this more physically in the language of [39, 48] by introducing a four-dimensional $N = 2$ nonlinear σ-model, in which case one identifies the quaternion coordinates u with hypermultiplets and the dual variables a and b with linear and hyper- multiplets respectively. However, we wish to stress that this interpretation is purely formal, since such a fictitious $N = 2$ theory need not (and will not) have any direct physical relevance in our situation.

We end by noting that the fiber $X(\vec{v}) = \vec{\pi}^{-1}(\vec{v})$ is given by:

$$X(\vec{v}) = \left\{u = [w_+, w_-] \mid (w_j^{(+)}, w_j^{(-)}) \text{ satisfy } (5.13)\right\}, \tag{5.15}$$

where the square brackets indicate that u is considered modulo the T^r action.

5.2.2 Degenerate fibers

The degenerations of the fibers of $\vec{\pi}$ can be described as follows [34]. For every toric hyperkähler generator ν_j, define a codimension three linear subspace $H_j = h_j \times h_j \times h_j$ of $\mathbb{R}^{3d} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$, where h_j is the following hyperplane in $\mathbb{R}^d$:

$$h_j = \{v \in \mathbb{R}^d \mid v \cdot \nu_j = 0\}\,. \tag{5.16}$$

Following the terminology of [34] we shall call H_j a *flat*; in terms of the real-complex coordinates $(a, b) \in \mathbb{R}^d \times \mathbb{C}^d$, it corresponds to the locus $a \cdot \nu_j = 0$, $b \cdot \nu_j = 0$, which by virtue of (5.13) amounts to $w_j^{(+)} = w_j^{(-)} = 0 \Leftrightarrow u_j = 0$. It thus follows from (5.15) that the preimage $\vec{\pi}^{-1}(H_j)$ is the sublocus of X defined by vanishing of the quaternion coordinate u_j:

$$X_j := \vec{\pi}^{-1}(H_j) = \{u \in X | u_j = 0\}\,. \tag{5.17}$$

It is clear that all flats have the origin of $\mathbb{R}^{3d}$ as a common point; the associated point in X is the apex $\vec{\pi}^{-1}(0)$, which corresponds to $u = 0$. On the other hand, two flats H_i, H_j intersect outside of the origin if and only if the associated hyperplanes h_i and h_j intersect outside the origin in $\mathbb{R}^d$. Since $X_i \cap X_j = \vec{\pi}^{-1}(H_i) \cap \vec{\pi}^{-1}(X_j) = \vec{\pi}^{-1}(H_i \cap H_j)$, we find that the loci X_i and X_j can intersect outside the apex of X if and only if h_i and h_j intersect outside the origin of $\mathbb{R}^d$.

Observation 1. Because $h_i \cap h_j$ has codimension at least two in $\mathbb{R}^d$, intersections outside the origin will always occur for $d > 2$. From this point of view, the case $d = 2$ of eight-dimensional toric hyperkähler cones is rather special; in that case, the hyperplanes h_j are simply lines through the origin in $\mathbb{R}^2$, and two such lines can intersect outside of the origin if and only if they coincide. As a consequence, two loci X_i and X_j (which in this case have real dimension four) will either coincide or intersect at the apex of X only. It is this special case which will be of interest in the remaining sections.

Observation 2. Recall that X can be viewed as an algebraic variety upon choosing the fist complex structure on $\mathbf{H}^n$. Since X_j are defined by the equations $w_j^{(+)} = w_j^{(-)} = 0$, each such locus is a sub-variety of X_j of complex codimension two.

The configuration of flats $H_1, \ldots, H_n$ can be used to describe the fixed points of the T^d action on X. Indeed, it is shown in [34] that the T^d-stabilizer of a point $u \in X$ is a subgroup $Stab_{T^d}(u)$ of T^d whose Lie algebra $stab_{T^d}(u)$ is given by:

$$stab_{T^d}(u) = \langle \{\nu_j | \vec{\pi}(u) \in H_j\} \rangle_{\mathbb{R}} \subset \mathbb{R}^d \,. \tag{5.18}$$

In this relation, $\langle \ldots \rangle_{\mathbb{R}}$ indicates the real linear span of the given set of vectors and $\mathbb{R}^d$ is viewed as the Lie algebra of T^d. Thus $stab_{T^d}(u)$ is spanned by those toric hyperkähler generators ν_j which have the property $\vec{\pi}(u) \cdot \nu_j = 0$, i.e. such that the associated flat H_j contains $\vec{\pi}(u)$. It is clear from this that all degenerate fibers sit above the flats. Above each flat H_j, a certain one-cycle of the T^d fiber collapses to zero length, with the Lie algebra of the collapsing S^1 lying in the direction ν_j. Above the intersection of k flats, the S^1 cycles associated with each flat collapse simultaneously.

5.3 The hyperkähler potential

The hyperkähler potential of the metric induced on X by the hyperkähler quotient construction can be obtained [34] by applying the general methods of [39]. To describe the result, it is convenient to use the coordinate $\vec{x} = \frac{\vec{v}}{2} = \frac{1}{2}\vec{\pi}(u)$. We have $\vec{x} = (x_1, x_2, x_3)$ with $x_s = (x_s^{(1)} \ldots x_s^{(d)}) \in \mathbb{R}^d$ for each $s = 1, 2, 3$. Then the real-complex coordinates a, b have the form

$$a = x_1 = (x_1^{(1)} \ldots x_1^{(d)}) = \frac{\pi_1(u)}{2}$$

and

$$b = x_3 + ix_2 = (x_3^{(1)} + ix_2^{(1)} \ldots x_3^{(d)} + ix_2^{(d)}) = \frac{\pi_3(u) + i\pi_2(u)}{2} \,.$$

We also consider coordinates $\phi^{(1)}, \ldots, \phi^{(d)}$ on the T^d fiber of $X \to \mathbb{R}^{3d}$, associated with the moment map components $\vec{\pi}^{(1)}, \ldots, \vec{\pi}^{(d)}$. It is convenient to define

$$r_k = 2\sqrt{(a \cdot \nu_k)^2 + (b \cdot \nu_k)(\bar{b} \cdot \nu_k)} = 2||\nu_j \cdot \vec{x}||_{\mathbb{R}^3} \,,$$

where $\nu_j \cdot \vec{x} = (\nu_j \cdot x_1, \nu_j \cdot x_2, \nu_j \cdot x_3)$ is a vector with three components. Note that r_k is proportional to the distance between $\vec{x}$ and H_k; in particular, the flat H_k corresponds to $r_k = 0$ (to understand this correctly, one must keep in mind that the flats have codimension *three* in $\mathbb{R}^{3d}$). It is shown in [34] that X admits the hyperkähler potential:

$$\mathcal{K} = \frac{1}{2}\sum_{k=1}^{n} r_k = \sum_{k=1}^{n} ||\nu_k \cdot \vec{x}||_{\mathbb{R}^3} \,, \tag{5.19}$$

which, as expected, is independent of the T^d fiber coordinates $\phi^{(1)}, \ldots, \phi^{(d)}$. Our normalization[14] is such that $\omega_A = i\partial_A\bar{\partial}_A\mathcal{K}$, where ω_A is the Kähler form associated with the

[14]This differs from the normalization of [34] by a factor of two.

compatible complex structure I_A. Note that a, b, $\vec{x}_j$ and r_k scale as α^2 under the homothetic action (5.1). Thus $\mathcal{K}$ has the α^2 scaling discussed in section 2, and the radial distance function $r = \mathcal{K}^{1/2}$ scales linearly. Using relations (5.13), one finds that $r_k = ||u_k||^2$ and thus:

$$\mathcal{K} = \frac{1}{2}\sum_k ||u_k||^2 = \frac{1}{2}||u||^2 . \tag{5.20}$$

The hyperkähler metric on X can be obtained [34] from (5.20) by using the general methods of [40].

5.4 Singularities

A toric hyperkähler cone $X = \mathbf{H}^n///_0 T^r = \mathcal{N}/T^r$ will generally have two types of singularities, namely those inherited from the variety $\mathcal{N} = \vec{\mu}^{-1}(0)$ and those due to fixed points of the T^r action. A singularity of $\mathcal{N}$ may become worse after taking the quotient, since the torus T^r may have a subgroup acting trivially on a singular locus of $\mathcal{N}$.

5.4.1 Good toric hyperkähler cones

It is possible to identify a class of toric hyperkähler cones for which $\mathcal{N}$ is smooth except at the origin. We show in appendix B that the following conditions are equivalent:

(a) All $d \times d$ minor determinants of the matrix G are nonzero. Equivalently, any d of the vectors $\nu_1, \ldots, \nu_n$ are linearly independent over $\mathbb{R}$ (and thus form a basis of $\mathbb{R}^d$),

(b) All $r \times r$ minor determinants of Q are nonzero,

and that, if they are satisfied, then the origin of $\mathbf{H}^n$ is the only singular point of $\mathcal{N}$.

Toric hyperkähler cones satisfying these conditions will be called *good*.

5.4.2 Singularities of good toric hyperkähler cones

Given a *good* toric hyperkähler cone X, the set $\mathcal{N}$ is smooth outside the origin and thus all singularities of $X - \{0\}$ arise from fixed points of the T^r action. On the other hand, (5.18) shows that such fixed points (and thus singularities of $X - \{0\}$) can only occur on the subloci $X_j = \vec{\pi}^{-1}(H_j)$ obtained by setting some quaternion coordinate u_j to zero. If u is a point of X, we let $V(u)$ be the set of indices j such that $u_j = 0$ and $N(u)$ be the set of indices j such that $u_j \neq 0$. This provides a partition of the index set $\{1, \ldots, n\}$, which characterizes the collection of flats which contains the point $\vec{\pi}(u)$. Indeed, one has $\vec{\pi}(u) \in H_j \Leftrightarrow u \in X_j = \vec{\pi}^{-1}(H_j)$ if and only if $j \in V(u)$. Therefore, the stabilizer (5.18) can be written:

$$stab_{T^d}(u) = \langle\{\nu_j | j \in V(u)\}\rangle_{\mathbb{R}} . \tag{5.21}$$

It is not hard to see (appendix B) that, for a *good* toric hyperkähler cone, the set $V(u)$ has at most $d-1$ elements (and thus $N(u)$ has at least $r+1$ elements), unless u coincides with the apex of X (in which case $V(u) = \{1, \ldots, n\}$). This happens because any d of the toric hyperkähler generators are linearly independent, which implies that no more than $d-1$ of the flats H_j can intersect outside of the origin in $\mathbb{R}^{3d}$.

It turns out that the partition $\{1, \ldots, n\} = V(u) \cup N(u)$ also characterizes the singularity type of X at the point u. Indeed, it is clear that u will be fixed by an element $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in T^r$ under the action (4.10) if and only if $\boldsymbol{\lambda}$ is a solution of the system:

$$\prod_{\alpha=1}^{r} \lambda^{q_j^{(\alpha)}} = 0 \qquad \text{for } j \in N(u)\,. \tag{5.22}$$

Let us assume that $u \neq 0$. Since the cone X is good and N contains at least $r+1$ elements, the r rows of exponents $q^{(\alpha)}$ $(\alpha = 1, \ldots, r)$ appearing in (5.22) are linearly independent. As discussed in appendix A, the solution set of such a system forms a multiplicative subgroup Γ_u of T^r, whose structure can be determined by computing the integral Smith form of the matrix Q_N obtained from Q by deleting all columns associated with indices belonging to the set $V(u)$. Using the fact that X is good, one can in fact show that the structure of Γ_u can also be determined from the integral Smith form of the matrix G_V obtained from G by deleting all columns associated with the index set $N(u)$. This result follows by chasing a certain diagram of lattices. We refer the reader to appendix A for the precise formulation and proof of these statements.

5.4.3 The eight-dimensional case

Let us consider the case of eight-dimensional toric hyperkähler cones, which are of interest for the remainder of this paper. In this case, one has $d = 2$ and $r = n - 2$. Let $X = \mathbf{H}^n///_0 T^{n-2}$ be such a cone. As remarked in Observation 1 of subsection 5.2.2, the case $d = 2$ is special due to dimension constraints which force the flats H_j to intersect only at the origin unless they coincide. Using this observation, it is not hard to see that the cone X will be good if and only if no two of the flats coincide. In this case, the flats intersect only at the apex, which means that the only degenerations allowed for the T^2 fibers of $X - \{0\}$ correspond to the collapse of a single cycle, a phenomenon which occurs along the four-dimensional loci X_j. Equivalently, at most one quaternion coordinate of a point u of $X - \{0\}$ can vanish (figure 7). By the results mentioned above, this means that the singularity type is constant above each flat (with the exception of the apex of X). On the locus $X_j - \{0\}$, one has a single vanishing coordinate u_j, which means that $V(u) = \{j\}$ and the matrix G_V coincides with the j^{th} column of G, i.e. with the toric hyperkähler generator ν_j. Therefore, the second description of singularities mentioned in the previous subsection becomes particularly simple. Combining these observations, one can show that the following statements are equivalent:

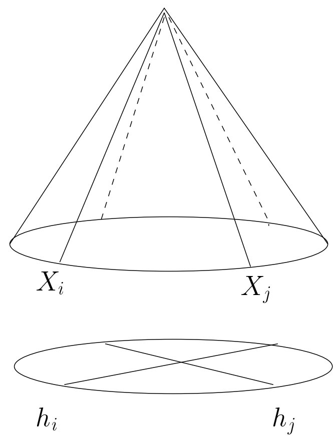

Figure 7: For a good, eight-dimensional toric hyperkähler cone, two four-dimensional subloci X_i and X_j can intersect only at the apex. Equivalently, two flats H_i and H_j and two lines h_i and h_j can intersect only at the origin. The figure shows two loci X_i, X_j and the associated lines $h_j, h_j \subset \mathbb{R}^2$.

(a) X is a good toric hyperkähler cone, i.e. any two of the toric hyperkähler generators $\nu_1, \ldots, \nu_n$ are linearly independent over $\mathbb{R}$. Equivalently, all $(n-2)\times(n-2)$ minor determinants of the $(n-2)\times n$ charge matrix Q are non-vanishing.

(b) No two of the three-dimensional flats $H_1, \ldots, H_n$ coincide in $\mathbb{R}^6$

(c) No two of the lines $h_1, \ldots, h_n$ coincide in $\mathbb{R}^2$.

In this case, the singularities of X can be described as follows:

1. All singularities of X lie in one of the four-dimensional loci $X_j = \{u \in X | u_j = 0\} = \pi^{-1}(H_j)$. Two such loci intersect at precisely one point, namely the apex of X.
2. The locus $X_j - \{0\}$ is smooth if and only if the associated toric hyperkähler generator $\nu_j \in \mathbb{Z}^2$ is a primitive vector.
3. If ν_j is not primitive, then each point on the locus $X_j - \{0\}$ is a $\mathbb{Z}_{m_j}$ quotient singularity of X, where m_j is the greatest common divisor of the coordinates of ν_j.

The formal proof of these statements can be found in appendix B. The singularity group $\mathbb{Z}_{m_j}$ appears as the multiplicative group of solutions $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-2}) \in T^{n-2}$ to the system (5.22) for $V(u) = \{j\}$:

$$\prod_{\alpha=1}^{n-2} \lambda^{q_k^{(\alpha)}} = 1 \qquad \text{for } k \neq j\,. \tag{5.23}$$

In appendix B, we show that such a solution will automatically also satisfy the equation:

$$\prod_{\alpha=1}^{n-2} \lambda^{q_j^{(\alpha)}} = e^{\frac{2\pi i}{m_j} s} \tag{5.24}$$

for some element $s \in \mathbb{Z}_{m_j}$. The isomorphism between Γ_j and $\mathbb{Z}_{m_j}$ is given by the map which takes a solution of (5.23) into the element s.

6. Quaternion-Kähler spaces and twistor spaces from toric hyperkähler cones

Given a toric hyperkähler cone $X = \mathbf{H}^n///_0 T^r$, the associated twistor space Y and quaternion-Kähler space M can be recovered as follows.

6.1 Construction of M as a conformal quotient

Let us consider the group $\mathbf{H}^* = \mathbf{H} - \{0\}$ of invertible quaternions, which acts on $\mathbf{H}^n$ through:

$$u_j \to u_j\, t^{-1}\,, \qquad t \in \mathbf{H}^*\,. \tag{6.1}$$

Since t acts from the right in (6.1), this commutes with the T^r action (4.10), and thus descends to an $\mathbf{H}^*$ action on the hyperkähler cone X.

The subgroup of $\mathbf{H}^*$ consisting of unit norm quaternions is the symplectic orthogonal group Sp(1), whose action (6.1) rotates the complex structures of $\mathbf{H}^n$; this descends to an action on X which rotates its complex structures and obviously preserves the hyperkähler potential (5.20). As discussed in section 3, restriction from X to a level set $\mathcal{K} = \zeta$ defines the associated 3-Sasaki space S, and the quotient:

$$M = S/\,\mathrm{Sp}(1)_{\mathrm{eff}} = \mathcal{K}^{-1}(\zeta)/\,\mathrm{Sp}(1)_{\mathrm{eff}} \tag{6.2}$$

is a quaternion-Kähler space of positive scalar curvature. This is the presentation of M as a conformal quotient [48, 31]. As mentioned in section 3, the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$ of Sp(1) may act trivially on X, so the effectively acting group is $\mathrm{Sp}(1)_{\mathrm{eff}} = \mathrm{Sp}(1)$ or $\mathrm{Sp}(1)/\mathbb{Z}_2 = \mathrm{SO}(3)$. A criterion for deciding when the $\mathbb{Z}_2$ subgroup acts trivially is given in subsection 5.4. below.

6.2 Description of M as a quaternionic quotient

Since our cone is toric hyperkähler, it is also possible to present M as a quaternionic quotient in the sense of [36]. For this, one considers the quotient of $\mathbf{H}^n$ through (6.1), which is the quaternion projective space $\mathbf{H}\mathbb{P}^{n-1}$. This is a quaternion-Kähler (but not hyperkähler) manifold. The quaternion projective space carries a quaternion analogue of the Fubini-Study metric:

$$ds^2 = \zeta\left[\frac{1}{||u||^2}\sum_{j=1}^{n} d\overline{u}_j du_j - \frac{1}{||u||^4}\sum_{j,k=1}^{n} \overline{u}^j du^j d\overline{u}^k u^k\right], \tag{6.3}$$

where the scale factor $\zeta > 0$ fixes the volume of $\mathbf{H}\mathbb{P}^{n-1}$. The T^r action (4.10) descends to an action on $\mathbf{H}\mathbb{P}^{n-1}$ which preserves its quaternion-Kähler structure. Then the work of [33, 36] implies that M can also be described as the quaternionic quotient:[15]

$$M := \mathbf{H}\mathbb{P}^{n-1}///T^r\,. \tag{6.4}$$

In this presentation, the quaternion structure of M is inherited from that of $\mathbf{H}\mathbb{P}^{n-1}$ by quaternionic reduction [36]. The scale of the resulting metric is fixed by the choice of ζ.

6.3 Description of the twistor space as a Kähler quotient

Returning to the action (6.1) on the quaternion affine space, let us pick the first complex structure $\mathbf{I}$ and write the elements of $\mathbf{H}^*$ in the form:

$$t = \alpha(\overline{\lambda} + \mathbf{j}\kappa)\,, \qquad \text{with} \quad \alpha = ||t|| > 0\,, \quad \lambda, \kappa \in \mathbb{C}, \quad |\lambda|^2 + |\kappa|^2 = 1\,. \tag{6.5}$$

Then (6.1) becomes:

$$\begin{aligned} w_k^{(+)} &\to \frac{1}{\alpha}\left(\lambda w_k^{(+)} + \kappa \overline{w}_k^{(-)}\right) \\ \overline{w}_k^{(-)} &\to \frac{1}{\alpha}\left(-\overline{\kappa} w_k^{(+)} + \overline{\lambda}\,\overline{w}_k^{(-)}\right). \end{aligned} \tag{6.6}$$

[15] Since $\mathbf{H}\mathbb{P}^{n-1}$ is only quaternion-Kähler, its reduction is only possible at level zero [36].

The subgroup of $\mathbf{H}^*$ consisting of unit norm quaternions $\tau = \overline{\lambda} + \mathbf{j}\kappa$ coincides with Sp(1). It acts on the vector $w_k := \begin{bmatrix} w_k^{(+)} \\ \overline{w}_k^{(-)} \end{bmatrix}$ through:

$$w_k \to A(\tau) w_k \, , \tag{6.7}$$

where $A(\tau)$ is the SU(2) matrix:

$$A(\tau) = \begin{bmatrix} \lambda & \kappa \\ -\overline{\kappa} & \overline{\lambda} \end{bmatrix} . \tag{6.8}$$

The map $\tau \to A(\tau)$ gives the standard group isomorphism $\mathrm{Sp}(1) \approx \mathrm{SU}(2)$. Since the vector w_k appearing in (6.7) contains the complex conjugate of $w_k^{(-)}$, this action rotates (i.e. acts transitively on) the complex structures.

Fixing $\kappa = 0$ gives a U(1) subgroup $\tau = \lambda$ of Sp(1) ($|\lambda| = 1$), which is identified with the diagonal U(1) subgroup of SU(2) given by $A(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$. This T^1 subgroup acts on $w^{(+)}$ and $w^{(-)}$ through:

$$w^{(\pm)} \to \lambda w^{(\pm)} \, , \tag{6.9}$$

and in particular it *does* preserve the complex structure $\mathbf{I}$ (this is why the Sp(1) orbit of the complex structures is $\mathrm{Sp}(1)/\mathrm{U}(1) = S^2$). The action (6.9) on $\mathbf{H}^n$ admits the *Kähler* moment map:

$$\mu_0(u) = \frac{1}{2} \sum_{j=1}^{n} (|w_j^{(+)}|^2 + |w_j^{(-)}|^2) = \frac{1}{2} \sum_{j=1}^{n} ||u_j||^2 \, . \tag{6.10}$$

The induced T^1 action on the hyperkähler cone preserves the first complex structure $\mathbf{I}$ of X and has a moment map induced from (6.10), which obviously coincides with the hyperkähler potential $\mathcal{K}$ of (5.20).

As explained in section 3, the twistor space is given by the quotient:

$$Y = \mathcal{K}^{-1}(\zeta)/T^1_{\text{eff}} = X//_{\zeta} T^1_{\text{eff}} \, , \tag{6.11}$$

where T^1_{eff} is the effectively acting subgroup of T^1.

Observation. Under the action (6.7), the projection $(\pi_r(u), \pi_c(u)) = (a, b)$ of subsection 5.2. transforms as:

$$\begin{aligned} a &\to (|\lambda|^2 - |\kappa|^2) a + 2\,\mathrm{Re}(\lambda \overline{\kappa} b) \\ b &\to |\lambda|^2 b - |\kappa|^2 \overline{b} - 2\lambda\kappa a \, . \end{aligned} \tag{6.12}$$

In particular, the element $\mathbf{j} \in \mathrm{Sp}(1)$ (which corresponds to $\lambda = 0$ and $\kappa = 1$) acts on complex coordinates as:

$$w_k^{(+)} \to \overline{w}_k^{(-)} \, , \qquad w_k^{(-)} \to -\overline{w}_k^{(+)} \tag{6.13}$$

and induces the transformations:

$$a \to -a \, , \quad b \to -\overline{b} \qquad \Leftrightarrow \qquad v_1 \to -v_1 \, , \quad v_2 \to v_2 \, , \quad v_3 \to -v_3 \, . \tag{6.14}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "\hat{q}^*"] \arrow[d, "s"'] & \mathbb{Z}^{2n} \arrow[r, "\hat{g}"] \arrow[d, "id"'] & \mathbb{Z}^{2d+r} \arrow[r] \arrow[d, "p"'] & 0 \\
0 \arrow[r] & \mathbb{Z}^{r+1} \arrow[r, "\tilde{q}^*"] & \mathbb{Z}^{2n} \arrow[r, "\tilde{g}"] & A \arrow[r] & 0
\end{tikzcd}$$

Figure 8: Exact sequences for the toric embedding of Y.

On the other hand, the fibers of $Y \to M$ are the $S^2 = \mathrm{Sp}(1)/\,\mathrm{U}(1)$ orbits of the induced $\mathrm{Sp}(1)$ action on Y. It is clear that (6.13) acts along these fibers, and therefore induces an involution of Y which commutes with the projection $Y \to M$. This is the so-called 'antipodal map' of the S^2 fibration $Y \to M$. Equations (6.14) give the projection of this involution through the T^d fibration map $\vec{\pi} : Y \to \mathbb{R}^{3d}$.

6.4 Embedding of the twistor space in a toric variety

The last equation in (6.11) can be used to embed Y in a toric variety. For this, remember from subsection 5.1. that $X = \mu_c^{-1}(0)//_0 T^r$ and notice that the T^1 action (6.9) preserves the level set $\mu_c^{-1}(0)$. Since $\mathcal{K}$ is induced by the moment map (6.10), we find that Y coincides with the Kähler quotient:

$$Y = \mathcal{Z}//T^{r+1}_{\text{eff}}\,, \tag{6.15}$$

where $\mathcal{Z} \in \mathbf{H}^n = \mathbb{C}^{2n}$ is the affine variety of subsection 5.1. and T^{r+1}_{eff} is the effectively acting subgroup of T^{r+1}. In this presentation, the first r reductions are performed at zero moment map levels, while the last is performed at positive level ζ. The metric induced by the reduction is the Kähler-Einstein metric of Y. Using the Kähler-quotient-holomorphic quotient correspondence, we obtain $Y = (\mathcal{Z} - \{0\})/(\mathbb{C}^*)^{r+1}$, which allows us to view Y as the intersection of quadrics (5.8) in the ambient toric variety $\mathbb{T} = \mathbb{S}//_\zeta\,\mathrm{U}(1) = (\mathbb{C}^{2n} - \{0\})/(\mathbb{C}^*)^{r+1}$. In the Kähler quotient description of $\mathbb{T}$, the first r quotients are performed at zero moment map levels, while the last quotient is performed at an arbitrary (but fixed) positive level.

The torus T^r maps into T^{r+1} according to the map $s : \mathbb{Z}^r \to \mathbb{Z}^{r+1}$ given by $s(v) = (v, 0)$. The T^{r+1} action on $\mathbb{C}^{2n}$ is described by the map $\tilde{q}^* : \mathbb{Z}^{r+1} \to \mathbb{Z}^{2n}$, whose transpose corresponds to the toric charge matrix $\tilde{Q}$ obtained by augmenting $\hat{Q}$ with an $(r+1)^{\text{th}}$ row:

$$\tilde{Q} = \begin{bmatrix} Q & -Q \\ 1\cdots1 & 1\cdots1 \end{bmatrix}. \tag{6.16}$$

The maps $\hat{q}^*, \tilde{q}^*$ and s satisfy:

$$\tilde{q}^* \circ s = \hat{q}^*\,. \tag{6.17}$$

The toric ambient space $\mathbb{T}$ is described by a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^{r+1} \xrightarrow{\tilde{q}^*} \mathbb{Z}^{2n} \xrightarrow{\tilde{g}} A \longrightarrow 0\,, \tag{6.18}$$

where the group A will generally contain torsion. This corresponds to the fact that the projectivising $\mathrm{U}(1)$ action need not be effective on $\mathbb{S}$.

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The situation is described by the commutative diagram of figure 8. Applying the 3-lemma gives a unique and surjective linear map $p : \mathbb{Z}^{2d+r} \to \mathbb{Z}^{2d+r-1}$ satisfying the constraint $p \circ \hat{g} = \tilde{g}$.

6.5 Quotient description of the 3-Sasaki space

The form (5.20) of the hyperkähler potential shows that the 3-Sasaki space S is the *3-Sasaki reduction* [27] of the sphere $S^{4n-1} = \{u \in \mathbf{H}^n \mid \frac{1}{2}||u||^2 = \zeta\}$ through the action of T^r:

$$S = [S^{4n-1} \cap \vec{\mu}^{-1}(0)]/T^r \,. \tag{6.19}$$

In fact, S^{4d-1} admits a 3-Sasaki structure determined by its hyperkähler cone $\mathbf{H}^n$, and the restriction of the hyperkähler moment map $\vec{\mu}$ to this sphere is a so-called *3-Sasakian moment map* [27]. Certain classes of torus reductions of 3-Sasakian spheres were studied in [28], though their singularities were not determined there.

6.6 On effectiveness of the $\mathrm{Sp}(1)$ and $\mathrm{U}(1)$ actions on X

In this subsection, we give a criterion for deciding when the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$ of $\mathrm{Sp}(1)$ (and of T^1) acts trivially on X. For this, consider the integral Smith form Q_{ismith} of the $r \times n$ charge matrix Q and matrices $U \in \mathrm{SL}(r, \mathbb{Z})$, $V \in \mathrm{SL}(n, \mathbb{Z})$ such that $Q = U^{-1} Q_{\mathrm{ismith}} V$. As explained in section 4, Q_{ismith} has the form $[I, 0]$, where I is the $r \times r$ identity matrix.

Since the $\mathbb{Z}_2$ subgroup acts through sign inversion of the quaternion coordinates u_j, it will have trivial action on X if and only if this transformation is realized by the T^r action, i.e. if and only if the system:

$$\prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_j^{(\alpha)}} = -1 \qquad \text{for all } j = 1, \ldots, n \,. \tag{6.20}$$

admits a solution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-2}) \in \mathrm{U}(1)^r$. This system is analyzed in appendix C, where we prove the following:

Proposition 1. *The $\mathbb{Z}_2$ subgroup $\{1, -1\}$ of $\mathrm{Sp}(1)$ acts trivially on X if and only if there exists $1 \leq m \leq r$ and $1 \leq \alpha_1 < \alpha_2 < \cdots \alpha_m \leq r$ such that all components of the n-vector w defined as the sum of the rows $\alpha_1, \ldots, \alpha_m$ of Q are odd. The indices α_k with this property are uniquely determined.*

This allows us to determine the effectively acting subgroups of $\mathrm{Sp}(1)$ and $\mathrm{U}(1)$ and is related to nonvanishing of the Marchiafava-Romani class of the quaternion-Kähler space M.

7. Twistor space singularities in the six-dimensional case

Throughout this section we consider a *good* eight dimensional toric hyperkähler cone $X = \mathbf{H}^n///T^{n-2}$, so that $d = 2$ and $r = n - 2$. We shall present a method for identifying the singularities of the six-dimensional twistor space $Y = X//_\zeta T^1$, where ζ is a fixed positive number. Our approach combines ideas from toric geometry with the description of X as a T^2 fibration over $\mathbb{R}^6$.

The basic idea is as follows. Since X is a good toric hyperkähler cone, its singularities outside the apex can be identified by the methods of section 5. These are the singularities along the loci $X_j - \{0\}$, and are the only singularities of X which can descend to Y, since the apex of X is removed when performing the projectivising $\mathrm{U}(1)$ quotient at a positive level. The corresponding loci in the twistor space are holomorphically embedded two-dimensional spheres $Y_j = X_j // \mathrm{U}(1)_{\mathrm{eff}}$, which turn out to coincide with certain fibers of the S^2 fibration $Y \to M$. The union Y_V of all Y_j will be called the *vertical locus*; we define this union to contain all components Y_j, even though some of them could in fact be smooth in Y. The singularity type along Y_j can be computed by toric methods. This singularity type may be enhanced with respect to X_j — in particular, a locus X_j which happens to be smooth in X may project to a sphere of $\mathbb{Z}_2$ singularities in Y.

A second class of singularities arises from smooth points of X which have nontrivial stabilizer under the projectivising action. Presenting X as a T^2 fibration over $\mathbb{R}^6$ as in section 5, the π-projection of such points must be invariant under the action of this stabilizer. This observation will allow us to extract a locus X_H which we define as the union of those T^2 fibers of X whose projection to $\mathbb{R}^6$ is fixed by a nontrivial subgroup of $\mathrm{U}(1)_{\mathrm{eff}}$. It will turn out that X_H consists of those points $u \in X$ having the property $\pi_c(u) = 0$, i.e. it is the union of the T^2 fibers which lie above the two-plane $b = 0$ in $\mathbb{R}^2 \times \mathbb{C}^2 = \mathbb{R}^6$. The real part π_r of the moment map π induces a T^2 fibration of X_H over this plane. When descending to the twistor space, one obtains the *horizontal locus* $Y_H = X_H // \mathrm{U}(1)$. Since this requires imposition of the moment map constraint $\frac{1}{2}\sum_{j=1}^{n} ||u_j||^2 = \zeta$, we shall find that Y_H is an S^1 fibration over a one-dimensional locus Δ lying inside the plane $b = 0$. It will turn out that Δ is a convex polygon with $2n$ vertices, which is entirely determined by ζ and by the matrix of generators G. We will call it the *characteristic polygon.* It is symmetric under reflection through the origin of the plane, and in particular contains the origin in its interior (figure 9).

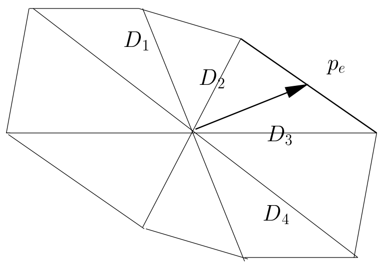

Figure 9: Shape of the characteristic polygon Δ for the case $n = 4$.

The S^1 fibers of Y_H degenerate to points above the vertices of Δ, which means that Y_H is a union of two-spheres Y_e associated with its edges e; two spheres Y_e and $Y_{e'}$ associated with adjacent edges intersect at a point Y_A in Y lying above their common vertex A. Each horizontal sphere Y_e turns out to be holomorphically embedded in Y, and is the lift of a locus lying in the base M through the S^2 fibration map $Y \to M$. Therefore, the intersection of Y_e with a given S^2 fiber consists of at most one point. Once again, Y_H is defined to contain all spheres Y_e, even though some of them may be smooth in Y. It will also turn out that two horizontal spheres $Y_e, Y_{e'}$ have the same projection on M if they are associated with opposite edges $e' = -e$ of Δ. In this case, they are either both smooth or singular in Y, with the same singularity type, which can be determined by a simple criterion. In fact, the sign inversion of $\mathbb{R}^2$ is covered by the antipodal map of the fibration $Y \to M$. Those diagonals D_j of Δ which connect opposite vertices (and thus pass through the origin) will be called *principal.* It will turn out that the restriction of π_r

to X_j can be used to present each vertical sphere Y_j as an S^1 fibration over the principal diagonal D_j; its S^1 fibers collapse above the opposite vertices which this diagonal connects. The corresponding points of Y_j coincide with the points of the horizontal locus which lie above these vertices. Each of these points corresponds to the intersection of Y_j with two horizontal spheres (figure 10).

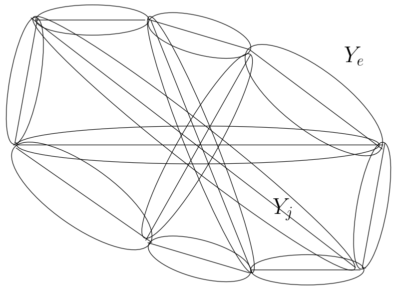

Figure 10: The distinguished locus (the union of the horizontal and vertical loci in the text) is an S^1 fibration over the edges and principal diagonals of Δ. The vertical spheres touch the horizontal spheres at single points, in spite of our inability to draw this in two dimensions.

7.1 The distinguished locus

Recall from section 5 that the hyperkähler moment map $\vec{\pi}$ presents X as a T^2 fibration over $\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The fiber above a point $\vec{v} = (v_1, v_2, v_3)$ (with v_j in $\mathbb{R}^2$) is given by the solutions of the system:

$$\frac{1}{2}\left(|w_j^{(+)}|^2 - |w_j^{(-)}|^2\right) = \nu_j \cdot a\,, \qquad w_j^{(+)} w_j^{(-)} = \nu_j \cdot b\,, \tag{7.1}$$

where $a = v_1/2 \in \mathbb{R}^2$ and $b = (v_3 + iv_2)/2 \in \mathbb{C}^2$. The variables $w_j^{(\pm)}$ are subject to identifications induced by the triholomorphic T^r action:

$$w_j^{(+)} \to e^{2\pi i \sum_{\alpha=1}^r q_j^{(\alpha)} \phi_\alpha} w_j^{(+)}\,, \qquad w_j^{(-)} \to e^{-2\pi i \sum_{\alpha=1}^r q_j^{(\alpha)} \phi_\alpha} w_j^{(-)}\,. \tag{7.2}$$

As explained in section 5, solutions to (7.1) will automatically satisfy the hyperkähler moment map constraints $\vec{\mu} = \vec{0}$.

The twistor space results from X by performing the projectivising quotient at a fixed level $\zeta > 0$, which amounts to imposing the moment map condition:

$$\mathcal{K}(w^{(+)}, w^{(-)}) = \frac{1}{2}\sum_{j=1}^n \left(|w_j^{(+)}|^2 + |w_j^{(-)}|^2\right) = \zeta \tag{7.3}$$

and quotienting by the action:

$$w_j^{(+)} \to \lambda w_j^{(+)}\,, \qquad w_j^{(-)} \to \lambda w_j^{(-)}\,, \tag{7.4}$$

where $\lambda = e^{2\pi i \phi}$. We let $\mathcal{M}_\zeta := \mathcal{K}^{-1}(\zeta) \subset X$ be the seven-dimensional locus defined by (7.3). Since $\mathcal{K}$ is invariant under the T^n action, the subspace $\mathcal{M}_\zeta$ is also a T^2 fibration, obtained by restricting the image of $\vec{\pi}$ from $\mathbb{R}^6$ to the 5-dimensional locus $\Sigma_\zeta := \vec{\pi}(\mathcal{M}_\zeta)$.

Remember that X is smooth outside $X_j = \vec{\pi}^{-1}(H_j) = \{u \in X | u_j = 0\}$. Hence singularities of Y can only occur on one of the loci $Y_j := (X_j \cap \mathcal{M}_\zeta)/\,\mathrm{U}(1)$ or at points outside these loci whose projection (a, b) to $\mathbb{R}^2 \times \mathbb{C}^2$ is stabilized by a nontrivial subgroup of $\mathrm{U}(1)_{\mathrm{eff}}$. Let $Y_V := \cup_{j=1}^n Y_j$. *We shall show that a point u lying outside Y_V can have a nontrivial stabilizer in* $\mathrm{U}(1)_{\mathrm{eff}}$ *only if* $b = 0$. The argument is as follows.

Using equations (7.1) and the fact that the vectors ν_j generate $\mathbb{R}^2$, it is easy to see that the projectivising U(1) action descends to a well defined action on Σ through the projection $\vec{\pi}$. From (7.1) and (7.4) it follows that a point $(\pi_r(u), \pi_c(u)) = (a, b)$ transforms as:

$$a \to a\,, \qquad b \to \lambda^2 b\,. \tag{7.5}$$

Hence the $\mathbb{Z}_2$ subgroup $\lambda = \pm 1$ of U(1) acts trivially on Σ. In particular, since the trivially acting subgroup G_0 of the projectivising U(1) must preserve (a, b), it follows that G_0 is a subgroup of this $\mathbb{Z}_2$, i.e. is either trivial or coincides with $\mathbb{Z}_2$. As explained in section 6, one has $G_0 = \mathbb{Z}_2$ if and only if the following system admits a solution $\lambda \in \mathrm{U}(1)^r$:

$$\prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_j^{(\alpha)}} = -1 \qquad \text{for all } j = 1, \ldots, n\,. \tag{7.6}$$

We distinguish two cases:

(a) If (7.6) has a solution, then $G_0 = \mathbb{Z}_2$ and the symmetry $u \to -u$ is not part of the effectively acting group $\mathrm{U}(1)_{\mathrm{eff}} = \mathrm{U}(1)/G_0 = \mathrm{U}(1)/\{-1, 1\}$. In this case, the transformation rule (7.5) shows that a point u can be stabilized by a nontrivial subgroup of $\mathrm{U}(1)_{\mathrm{eff}}$ only if $b = 0$.

(b) If (7.6) has no solutions, then G_0 is the trivial group, and the projectivising U(1) acts effectively on X. Let us assume that $b \neq 0$ (otherwise, there is nothing left to prove). In this case, relation (7.5) shows that the U(1) stabilizer must be a subgroup of $\{-1, 1\}$. This stabilizer is nontrivial precisely when there exists a $\mathrm{U}(1)^r$ transformation which implements the sign inversion $w_k^{(\pm)} \to -w_k^{(\pm)}$ on all complex coordinates of u. To show that the stabilizer is trivial, we proceed in two steps:

($b1$) Show that at most one complex coordinate of u can vanish.

To understand why, let us assume that two complex coordinates of u equal zero. Since u does not belong to $Y_V = \cup_{j=1}^n Y_j$, we cannot have $w_k^{(+)} = w_k^{(-)} = 0 \Leftrightarrow u_k = 0$, since this would imply $u \in Y_k$. Therefore, we must have $w_j^{(+)} = w_k^{(+)} = 0$ or $w_j^{(-)} = w_k^{(-)}$ for some $k \neq j$. In both cases, equations (7.1) give $\nu_j \cdot b = \nu_k \cdot b = 0$, which implies[16] $b = 0$, thereby contradicting our assumption.

($b2$) Show that no $\mathrm{U}(1)^r$ transformation can implement the sign inversion $w_i^{(\pm)} \to -w_i^{(\pm)}$.

According to ($b1$), only one of the complex coordinates of u can vanish. Without loss of generality, we can assume that $w_k^{(+)} = 0$. In this case, a $\mathrm{U}(1)^r$ transformation implementing the desired sign inversion exists if an only if the system:

$$\begin{aligned} \prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_j^{(\alpha)}} &= -1 \qquad \text{for all } j \neq k \\ \prod_{\alpha=1}^{n-2} \lambda_\alpha^{-q_j^{(\alpha)}} &= -1 \qquad \text{for all } j = 1, \ldots, n \end{aligned} \tag{7.7}$$

[16] Since we assume that the toric hyperkähler cone is good, the two-vectors ν_j and ν_k are linearly independent.

admits a solution $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathrm{U}(1)^r$. Now, it is clear that the second set of equations in this system implies the first, which means that the entire system is equivalent with (7.6), which has no solutions by the hypothesis of case (b). It follows that ($b2$) holds. Combining everything, we see once again that the $\mathrm{U}(1)$ stabilizer of u must be trivial unless $b = 0$. This finishes the proof of our claim.

Let us define Y_H to be the set of points u in Y for which $\pi_c(u) = 0 \Leftrightarrow b = 0$. According to the discussion above, a singular point of Y must belong to one of the loci Y_V or Y_H. We conclude that *all* singularities of Y lie along the *distinguished locus*:

$$Y_D := Y_V \cup Y_H \subset Y\,, \tag{7.8}$$

where $Y_V := \cup_{j=1}^n Y_j$ with $Y_j = \{u \in Y | u_j = 0\} = (X_j \cap \mathcal{M}_\zeta)/T^1$ and $Y_H := X_H/\,\mathrm{U}(1)$ where $X_H = \vec{\pi}^{-1}(\{(a,b) \in \Sigma | b = 0\})$.

7.2 Geometry of the component Y_H and the characteristic polygon Δ

To characterize Y_H, consider equations (7.1) for $b = 0$:

$$\frac{1}{2}(|w_j^{(+)}|^2 - |w_j^{(-)}|^2) = \nu_j \cdot a\,, \qquad w_j^{(+)} w_j^{(-)} = 0\,. \tag{7.9}$$

Since the second set of conditions requires either $w_j^{(+)} = 0$ or $w_j^{(-)} = 0$ for each j, we have 2^n possible branches $X_\epsilon \subset X_H$, parameterized by a 'sign vector' $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_j = \pm 1$. The branch X_ϵ is defined by choosing the solutions $w^{(-\epsilon_j)} = 0$ for the second set of equations in (7.9). Thus X_ϵ is given by the constraints:

$$w_j^{(-\epsilon_j)} = 0\,, \qquad \frac{1}{2}|w_j^{(\epsilon_j)}|^2 = \epsilon_j \nu_j \cdot a\,, \qquad \frac{1}{2}\sum_{j=1}^n |w_j^{(\epsilon_j)}|^2 = \zeta \tag{7.10}$$

and the action:

$$w_j^{(\epsilon_j)} \to e^{2\pi i \epsilon_j \sum_{\alpha=1}^r q_j^{(\alpha)} \phi_\alpha} w_j^{(\epsilon_j)}\,. \tag{7.11}$$

The second equation in (7.10) requires $\epsilon_j \nu_j \cdot a \geq 0$ for all $j = 1, \ldots, n$.

The third constraint in (7.10) results from the moment map condition (7.3). When combined with the second equation in (7.10), it becomes:

$$\sum_{j=1}^n |\nu_j \cdot a| = \zeta\,, \tag{7.12}$$

a condition for a whose solution set forms the union of edges of a convex polygon Δ in $\mathbb{R}^2$. Since (7.12) is invariant under the sign inversion $a \to -a$, this polygon is symmetric with respect to the origin.

The map $\vec{\pi}$ presents X_ϵ as a T^2 fibration over the following subset of Δ:

$$\Delta_\epsilon = \vec{\pi}(X_\epsilon) = \{a \in \Delta | \epsilon_j \nu_j \cdot a \geq 0\}\,. \tag{7.13}$$

We have $\Delta = \cup_\epsilon \Delta_\epsilon$ and $X_H = \cup_\epsilon X_\epsilon$. Moreover, X_ϵ will be non-void precisely when the system of equations:

$$\epsilon_j \nu_j \cdot a \geq 0\,, \qquad \sum_{j=1}^n |\nu_j \cdot a| = \zeta \tag{7.14}$$

admits a solution, in which case Δ_ϵ coincides with an edge of Δ. Thus one can index the *non-void* components among X_ϵ by the edges e of Δ: $X_\epsilon = X_e$ for some edge e, if $X_\epsilon \neq \emptyset$. Given an edge e, the associated signs ϵ_j are obtained as follows. If p_e is a vector lying in the interior of e (for example the middle point of e), we let $\epsilon_j(e)$ be the sign of the scalar product $\nu_j \cdot p_e$. Then $X_e = X_{\epsilon(e)}$. Thus X_e is the sublocus in X which corresponds to vanishing of the variables $w_j^{(-\epsilon_j(e))}$; in particular, this shows that X_e and Y_e are complex subvarieties of X and Y. Eliminating the void branches gives $X_H = \cup_e X_e$.

We conclude that X_H is obtained by restricting to those T^2 fibers of X which lie above the edges of the polygon Δ (note that this automatically implements the U(1) moment map constraint $\mathcal{K} = \zeta$). Since $b = 0$ on each X_e, equations (7.5) show that the projectivising U(1) action is fiberwise along this set. It follows that $Y_e = X_e/\,\mathrm{U}(1)$ is an S^1 fibrations over the edge e of Δ.

It is easy to see from (7.12) that the vertices of Δ lie along the lines $h_1, \ldots, h_n \subset \mathbb{R}^2$ given by equations (5.16): $h_j = \{v \in \mathbb{R}^2 | v \cdot \nu_j = 0\}$. Each such line contains precisely two vertices, one on each side of the origin of $\mathbb{R}^2$; these are mapped into each other by the sign inversion $a \to -a$. We shall let D_j denote the diagonal of Δ lying along the line h_j; such diagonals will be called *principal*.

Since $a = v_1/2$ and since $b = (v_3 + iv_2)/2$ vanishes along the horizontal locus, we find that h_j coincides with the intersection of the flat $H_j = h_j \times h_j \times h_j$ with the two-plane in $\mathbb{R}^6$ given by the first $\mathbb{R}^2$ factor in the decomposition $\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The discussion of section 3 shows that the T^2 fiber of X degenerates to a circle above each flat. In particular, the T^2 fibers of X above the edges of Δ degenerate to circles above each vertex (since the vertices of Δ lie on the lines $h_j \times 0 \times 0 \subset H_j$). Upon performing the projectivising quotient, this implies that the S^1 fibers of $Y_e \to e$ degenerate to points above the vertices. It follows that each locus Y_e is a two-sphere. The spheres associated with adjacent edges intersect at a point corresponding to their common vertex.

7.3 Geometry of the component Y_V

The subspaces $Y_k = \{u \in Y | u_k = 0\} = \{w \in Y | w_k^{(+)} = w_k^{(-)} = 0\}$ are one-dimensional complex subvarieties of Y. Since X_j are invariant under the SU(2) action induced by (6.1), the loci Y_j must correspond to the $\mathrm{SU}(2)/\,\mathrm{U}(1) = S^2$ orbits of the induced SU(2) action on Y; therefore, each Y_j is a fiber of the S^2 fibration $Y \to M$. In particular, Y_j are rational curves in the twistor space.

To see directly why Y_k is a two-sphere, notice that substituting $w_k^{(+)} = w_k^{(-)} = 0$ in equations (7.1) implies $\nu_k \cdot a = \nu_k \cdot b = 0 \Leftrightarrow \nu_k \cdot v = 0$, which forces the vector $v = (2a, 2\,\mathrm{Im}(b), 2\,\mathrm{Re}(b))$ to lie in the 3-dimensional subspace $H_k \subset \mathbb{R}^6$. The condition $(a, b) \in \Sigma_\zeta$ further constrains v to lie on a locus $\sigma_k \subset H_k$, which is topologically a two-sphere.

The value $b = 0 \Leftrightarrow v_2 = v_3 = 0$ gives two opposite points on this sphere (which one can take to be the north and south pole), which correspond to opposite vertices of the polygon Δ. Since $\nu_k \cdot a$ vanishes on our locus, the real component $\pi_r : X \to \mathbb{R}^2$ of the moment map descends to a projection of $X'_k := X_k \cap \mathcal{M}_\zeta$ onto the principal diagonal D_k. The locus $X'_k \subset X$ is an S^1 fibration over σ_k, since the generic T^2 fiber of $X \to \mathbb{R}^6$ is collapsed to a circle for $u_k = 0$ (figure 11). According to (7.5), the projectivising U(1) action on this locus fixes the value of $v_1 = 2a$ while rotating the vector $(v_2, v_3) = 2(\mathrm{Im}\, b, \mathrm{Re}\, b)$ in the two-plane defined by this value of v_1:

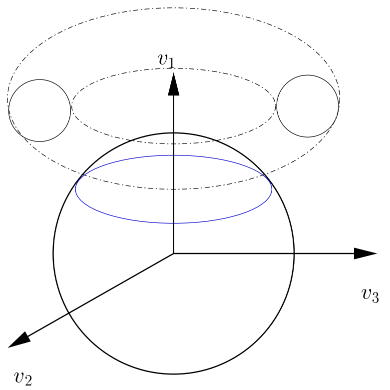

Figure 11: The locus $X'_k = X_K \cap \mathcal{M}_\zeta$ is an S^1 fibration over the two-sphere σ_k. The figure shows σ_k and the two-torus obtained by restricting to those S^1 fibers of X'_k which lie above one of the circles C_{v_1}.

$$\begin{aligned} v_1 &\to v_1 \\ v_2 &\to v_2 \cos(2\alpha) + v_3 \sin(2\alpha) \\ v_3 &\to -v_2 \sin(2\alpha) + v_3 \cos(2\alpha)\,, \end{aligned} \tag{7.15}$$

where we took $\lambda = e^{2i\alpha}$. The orbits are circles $C_{v_1} \subset \sigma_k$ lying in the plane defined by v_1 (figure 11). Due to the square in the second transformation (7.5), each orbit covers such a circle *twice*. In particular, the element $\lambda = -1 \Leftrightarrow \alpha = \pi$ effects a full rotation along the circle C_{v_1}. Since v_1 is invariant under this action, we find that $\pi_r = v_1$ descends to a projection of $Y_k = X'_k / \mathrm{U}(1)_{\mathrm{eff}}$ onto D_k.

The full projectivising U(1) action on the locus X'_k identifies the S^1 fibers of $X'_k \to \sigma_k$ sitting above the circles $C_{v_1} \subset \sigma_k$. The precise projection of this action on the S^1 fiber depends on the restriction of the transformations (7.2) to the locus X_k. There are two possibilities in this regard:

(*a*) $\mathrm{U}(1)_{\mathrm{eff}} = \mathrm{U}(1)$ and the $\mathbb{Z}_2$ subgroup generated by $\lambda = -1$ acts non-trivially on the S^1 fiber of $X'_k \to \sigma_k$.

In this case, one must go twice around the circle $C_{v_1} \subset \sigma_k$ in order to come back to the same point in the S^1 fiber. The projectivising U(1) quotient gives a copy of $\mathbb{RP}^1 = S^1$ fibered over the segment $\pi_r(X'_k) = D_k$; thus the induced map $\pi_r : Y_k \to D_k$ is an S^1 fibration. Since the circle C_{v_1} collapses to zero size at the poles of σ_k (which correspond to the vertices of Δ connected by D_k), we find the the S^1 fiber of Y_k collapses to a point above the endpoints of D_k. In particular, Y_k is a two-sphere.

(*b*) $\mathrm{U}(1)_{\mathrm{eff}} = \mathrm{U}(1)/\mathbb{Z}_2$ or $\mathrm{U}(1)_{\mathrm{eff}} = \mathrm{U}(1)$ and the $\mathbb{Z}_2$ subgroup of U(1) acts trivially in the direction of the S^1 fiber.

In this case, one comes back to the same point in the fiber after going once around the circle $C_{v_1} \subset \sigma_k$. Once again, the projectivising U(1) quotient gives an S^1 fibration $\pi_r : Y_k \to D_k$, whose fibers collapse above the endpoints of D_k.

We conclude that each locus Y_k is a two-sphere, which the map induced by π_r presents as an S^1 fibration over D_k. Such a sphere has two distinguished points, namely those points sitting above the opposite vertices of Δ connected by the principal diagonal D_k. Each distinguished point corresponds to $b = 0$ and therefore is shared by Y_k and two adjacent spheres belonging to the horizontal locus.

7.4 Relation with the S^2 fibration of Y over the quaternion-Kähler base

As discussed above, each vertical sphere Y_j is a fiber of $Y \to M$. For the horizontal locus Y_H, we have $b = 0$ and the SU(2) action (6.7) induces the following transformations (6.12):

$$\begin{aligned} a &\to (|\lambda|^2 - |\kappa|^2)a \\ b &\to -2\lambda\kappa a\,. \end{aligned} \tag{7.16}$$

Since $a \neq 0$, the second equation shows that the SU(2) orbit can touch Y_H at another point only if $\kappa = 0$ or $\lambda = 0$. The first case corresponds to the diagonal U(1) subgroup $A = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$ (with $|\lambda| = 1$), which acts trivially on the twistor space (since Y is a quotient of $\mathcal{K}^{-1}(\zeta) \subset X$ through this subgroup). The case $\lambda = 0$ gives $A = \begin{bmatrix} 0 & \kappa \\ -\overline{\kappa} & 0 \end{bmatrix}$, with $|\kappa| = 1$. In this case, the transformations (7.16) reduce to $a \to -a$ and $b = 0$ =fixed, while equation (6.7) gives:

$$w_k^{(+)} \to \kappa \overline{w}_k^{(-)}\,, \qquad w_k^{(-)} \to -\kappa \overline{w}_k^{(+)}\,. \tag{7.17}$$

These transformations obviously map the locus Y_e (defined by the equations $w_k^{(-\epsilon_k(e))} = 0$) into the locus Y_{-e} (defined by the equations[17] $w_k^{(+\epsilon_k(e))} = 0$). In particular, the antipodal map (which corresponds to the element $\mathbf{j} \in \mathrm{Sp}(1)$, for which $\lambda = 0$ and $\kappa = 1$) maps Y_e into Y_{-e} while taking a into $-a$. Since Sp(1) acts isometrically on Y, it follows that the singularity types of Y along Y_e and Y_{-e} must coincide. In fact, relation (6.14) shows that the antipodal map covers the sign inversion $a \to -a$ through the projection map $\pi_r : Y \to \mathbb{R}^2$; this relation between the antipodal map and sign inversion is valid on the entire distinguished locus $Y_D = Y_H \cup Y_V$.

The observations made above show that each sphere Y_e intersects the $\mathrm{Sp}(1)/\mathrm{U}(1) = S^2$ orbit of Sp(1) in precisely one point. Therefore, such spheres are horizontal with respect to the fibration $Y \to M$ (i.e. are lifts of spheres in the ESD base M).

The entire construction can be summarized as follows. One has a system of n spheres in M, described by the polygon Δ_M obtained from Δ upon quotienting through the sign inversion of $\mathbb{R}^2$ (figure 12). Each sphere in M corresponds to an edge of Δ_M, and two spheres intersect (at a single point) precisely when the associated edges of Δ_M touch each other at a vertex. Every such sphere has two lifts Y_e and Y_{-e} (related by the antipodal map) through the fibration $Y \to M$. These lifts correspond to those opposite edges e and $-e$ of Δ which lie above the associated edge of Δ_M. The collection of all such lifts gives the locus Y_H. The locus Y_V is the collection of S^2 fibers of $Y \to M$ which lie above the vertices of Δ_M. Comparing with the results of [28, 24] and [24], we see that the ESD space M is a T^2 fibration over the compact convex polytope bounded by Δ_M; the T^2 fibers collapse to circles above the edges of Δ_M and to points above its vertices. The polygon Δ_M can be identified with the polygon extracted in [28] by different methods. It is also homeomorphic with the boundary of the (compactified) hyperbolic plane appearing in [24].

[17] Notice that $\epsilon_k(-e) = -\epsilon_k(e)$.

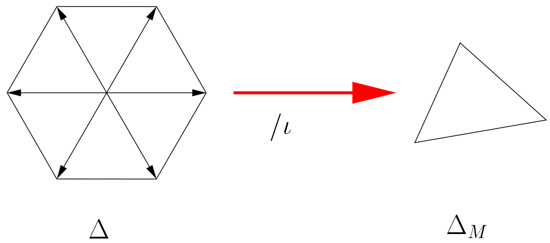

Figure 12: Dividing Δ through the sign inversion $\iota : a \to -a$ gives the polygon Δ_M. The figure shows the case $n = 3$.

7.5 Singularities along Y_H

Let us fix an edge e of Δ and consider the non-void branch $X_e \subset X$ associated with this edge. If the projectivising group[18] $\mathrm{U}(1)_{\text{eff}}$ acts effectively on X_e, then the associated two-sphere $Y_e = X_e/\mathrm{U}(1)_{\text{eff}}$ is a smooth locus in Y, except possibly for its two points lying above the endpoints of e. If this action is not effective, then there will be a discrete subgroup Γ_e of $\mathrm{U}(1)_{\text{eff}}$ which fixes every point of X_e; then Y_e consists of orbifold singularities of type Γ_e, with possible enhancement of the orbifold group at the endpoints of e. The group Γ_e can be identified through the following argument, a more formal version of which can be found in appendix D.

Let $\epsilon(e) = (\epsilon_1(e), \ldots, \epsilon_n(e))$ be the sign vector associated to the edge e as explained in subsection 7.1. Then X_e consists of the tori realized by equations (7.10) with the T^r identifications (7.11). Since the complex coordinates $w_j^{(-\epsilon_j(e))}$ vanish on X_e, the $(n-2)\times n$ charge matrix Q_e of the action induced by (7.11) on the non-vanishing coordinates $w_j^{(\epsilon_j(e))}$ is obtained from Q by changing the signs of its columns: the j^{th} column of Q is multiplied by $\epsilon_j(e)$. We have an exact sequence:

$$0 \longrightarrow \mathbb{Z}^{n-2} \xrightarrow{q_e^*} \mathbb{Z}^n \xrightarrow{g_e} \mathbb{Z}^2 \longrightarrow 0\,, \tag{7.18}$$

where q_e^* is the map defined by the transpose Q_e^t, while g_e is the map whose matrix G_e is obtained from G by multiplying its columns with the signs $\epsilon_j(e)$. This amounts to replacing the hyperkähler toric generators $\nu_j \in \mathbb{Z}^2$ with the vectors $\epsilon_j(e)\nu_j$.

Observation. The middle term of (7.18) has the following meaning. Consider the 'toric' diagonal T^{2n} action:

$$w_j^{(+)} \to \Lambda_j w_j^{(+)}\,, \qquad w_j^{(-)} \to \Lambda_{j+n} w_j^{(-)} \tag{7.19}$$

which is relevant to the description of X as an intersection of quadrics in a toric variety (subsection 5.1). Restricting this action to the non-vanishing coordinates $w_j^{(\epsilon_j(e))}$ on the locus X_e, we obtain:

$$w_j^{(\epsilon_j(e))} \to \lambda_j w_j^{(\epsilon_j(e))}\,, \tag{7.20}$$

where $\lambda_j = \Lambda_{j+(1-\epsilon_j(e))\frac{n}{2}}$. Tensoring the middle term of (7.18) with $\mathrm{U}(1)$ gives a torus T^n, which we let act on $w_j^{(\epsilon_j(e))}$ according to (7.20). This allows us to absorb the signs $\epsilon_j(e)$ into the definition of g_e.

[18] Recall that $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)$ or $\mathrm{U}(1)/Z_2$ is the group acting effectively *on* X.

The projectivising U(1) acts as follows on the locus X_e:

$$w_j^{(\epsilon_j)} \to e^{2\pi i\phi} w_j^{(\epsilon_j)} . \tag{7.21}$$

This action corresponds to a map form $T^1 = S^1$ to T^n, described by the lattice map $\gamma : \mathbb{Z} \to \mathbb{Z}^n$ which takes the generator 1 of $\mathbb{Z}$ into the vector $\left[\begin{smallmatrix} 1 \\ \cdots \\ 1 \end{smallmatrix}\right]$. The induced U(1) action on the T^2 fiber T^n/T^{n-2} of X_e is described by the composite map $\alpha_e = g_e \circ \gamma : \mathbb{Z} \to \mathbb{Z}^2$, which takes $1 \in \mathbb{Z}$ into the two-vector:

$$\nu_e := \sum_{j=1}^{n} \epsilon_j(e)\nu_j . \tag{7.22}$$

As explained in appendix D, the vector ν_e cannot vanish, which means the that the map α_e is injective. Therefore, we have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha_e} \mathbb{Z}^2 \xrightarrow{\beta_e} A_e \longrightarrow 0 . \tag{7.23}$$

Let us first assume that $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)$, so that the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$ acts nontrivially on X. In this case, the group A_e will contain torsion if and only if Γ_e is nontrivial. From the results of appendix A, we have $\Gamma_e = \mathbb{Z}/\mathbb{Z}\nu_e = \mathbb{Z}_{m_e}$, where m_e is the greatest common divisor of the coordinates of ν_e. In particular, the locus Y_e is smooth if and only if the vector ν_e is primitive. The embedding of Γ_e into T^n takes the generator of $\mathbb{Z}_{m_e}$ into the element $\lambda = (e^{\frac{2\pi i}{m_e}}, \ldots, e^{\frac{2\pi i}{m_e}})$ of T^n. When combined with (7.4), this describes the action of Γ_e on the coordinates $w_j^{(-\epsilon_j(e))}$ transverse to the locus Y_e:

$$w_j^{(-\epsilon_j(e))} \to e^{\frac{2\pi i}{m_e}} w_j^{(-\epsilon_j(e))} . \tag{7.24}$$

If the $\mathbb{Z}_2$ subgroup of the projectivising U(1) acts trivially on X, then one has $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)/\mathbb{Z}_2$ and the singularity group of Y along Y_e is the quotient $\mathbb{Z}_{m_e}/\mathbb{Z}_2$, where m_e is determined as above.

The group Γ_e can also be computed as the trivially acting subgroup of $T^{n-1} = T^{n-2} \times T^1$ on the locus $w_1^{(-\epsilon_1(e))} = \ldots = w_n^{(-\epsilon_n(e))} = 0$ in X. This is related to the 'toric' approach discussed below in subsection 7.9. The equivalence of the two methods is explained in appendix D.

7.6 Singularities along Y_V

Let us fix a component Y_j of Y_V. To identify the singularity type along Y_j, we must find the trivially acting subgroup of the restriction of the $\mathrm{U}(1)_{\text{eff}}^{n-1}$ action to the locus $w_j^{(+)} = w_j^{(-)} = 0$. This gives the system of equations:

$$\lambda \prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_k^{(\alpha)}} = \lambda \prod_{\alpha=1}^{n-2} \lambda_\alpha^{-q_k^{(\alpha)}} = 1 \qquad \text{for } k \neq j , \tag{7.25}$$

which encodes the condition that the non-vanishing coordinates $w_k^{(\pm)}$ $(k \neq j)$ must be fixed by the $\mathrm{U}(1)_{\text{eff}}^{n-1}$ action. Here $\lambda, \lambda_1, \ldots, \lambda_{n-2} \in \mathrm{U}(1)$. Since an element $(\lambda, \lambda_1, \ldots, \lambda_{n-2})$

which acts trivially on our locus must fix the vectors a, b, we find the constraint $\lambda^2 = 1$ i.e. $\lambda \in \{-1, 1\}$ (see equation 7.5). Hence it suffices to consider only these values of λ in the system (7.25). Following the discussion of subsection 7.1., we distinguish the following cases:

(a) If $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)/\mathbb{Z}_2$, then the $\mathbb{Z}_2$ subgroup of the projectivising U(1) acts trivially on X and is eliminated when constructing an effective action. Therefore, the system (7.25) reduces to:

$$\prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_k^{(\alpha)}} = 1 \qquad \text{for } k \neq j\,. \tag{7.26}$$

This coincides with the defining system (5.23) for the singularity group of the cone X along $X_j - \{0\}$. According to the results of subsection 5.4.3, the multiplicative group of solutions to this system is isomorphic with $\mathbb{Z}_{m_j}$. Therefore, the singularity group of Y along Y_j coincides with the singularity group $\mathbb{Z}_{m_j}$ of X along $X_j - \{0\}$.

(b) If $\mathrm{U}(1)_{\text{eff}} = \mathrm{U}(1)$, then the $\mathbb{Z}_2$ subgroup of the projectivising U(1) acts nontrivially on X, and we must consider both values $\lambda = 1$ and $\lambda = -1$. In this case, (7.25) reduces to:

$$\begin{gathered}\prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_k^{(\alpha)}} = 1 \qquad \text{for } k \neq j \\ \text{or} \\ \prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_k^{(\alpha)}} = -1 \qquad \text{for } k \neq j\,.\end{gathered} \tag{7.27}$$

In lattice language, we have the map $q_j^* : \mathbb{Z}^{n-2} \to \mathbb{Z}^{n-1}$ whose transpose matrix is obtained from Q by deleting the j^{th} column. This induces a map of tori $q_j^* : T^{n-2} \to T^{n-1}$, which we denote by the same letter. The group Γ_j of solutions to (7.27) is the subgroup $(q_j^*)^{-1}(im q_j^* \cap \{-1, 1\})$, where $\{-1, 1\}$ is the multiplicative $\mathbb{Z}_2$ subgroup of T^{n-1} formed by the elements $1 := (1, \dots, 1)$ and $-1 := (-1, \dots, -1)$. The kernel of q_j^* is the group of solutions to the first set of equations in (7.27), which by the results of subsection 5.4.3 is the singularity group $\mathbb{Z}_{m_j}$ of X along $X_j - \{0\}$. We distinguish the following cases:

(bI) $-1 \notin im q_j^*$, i.e. the second system of equations in (7.27) does not admit solutions. In this case, we have $im q_j^* \cap \{-1, 1\} = \{1\}$, and $\Gamma_j = \ker(q_j^* : T^{n-2} \to T^{n-1})$ coincides with the singularity group $\mathbb{Z}_{m_j}$ of X along X_j.

(bII) $-1 \in im q_j^*$, i.e. the second system of equations in (7.27) admits solutions. In this case, we have an exact sequence:

$$1 \to \mathbb{Z}_{m_j} \to \Gamma_j \stackrel{q_j^*}{\to} \mathbb{Z}_2 \to 1\,, \tag{7.28}$$

where the group on the right is the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$ of T^{n-1}. This shows that Γ_j is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{m_j}$. Since $Ext^1(\mathbb{Z}_2, \mathbb{Z}_{m_j}) = \mathbb{Z}_c$ [30], where c is the greatest common divisor of 2 and m_j, we distinguish the following possibilities:

(bII.1) m_j is odd. In this case, $Ext^1(\mathbb{Z}_2, \mathbb{Z}_{m_j}) = 0$ and (7.28) must be the trivial extension $\Gamma_j = \mathbb{Z}_2 \times \mathbb{Z}_{m_j}$. Since m_j is odd, we also have $\mathbb{Z}_2 \times \mathbb{Z}_{m_j} \approx \mathbb{Z}_{2m_j}$, so that $\Gamma_j = \mathbb{Z}_{2m_j}$. The isomorphism $\mathbb{Z}_2 \times \mathbb{Z}_{m_j} \to \mathbb{Z}_{2m_j}$ maps an element (α, u) into the element $2u + m_j\alpha$, so that $\{0\} \times \mathbb{Z}_{m_j}$ is mapped into the subgroup $\{0, 2, 4, \ldots, 2(m_j - 1)\}$ of $\mathbb{Z}_{2m_j}$; this corresponds to solutions of the first system in (7.27).
The order two element m_j of $\mathbb{Z}_{2m_j}$ does not belong to this subgroup (since m_j is odd). Hence this element must correspond to an element $\boldsymbol{\lambda}$ which satisfies the second system in (7.25). Since m_j has order two in $\mathbb{Z}_{2m_j}$, we must have $\boldsymbol{\lambda}^2 = 1$, so that $\lambda_\alpha = \pm 1$. It follows that the second system in (7.27) must admit a solution with $\lambda_\alpha \in \{-1, 1\}$, provided that it admits any solutions at all. For such values of λ_α, this system reduces to the condition that the sum of those rows α of Q_j for which $\lambda_\alpha = -1$ is a vector all of whose entries are odd. Hence if m_j is odd, then a necessary and sufficient condition for the second system in (7.27) to admit solutions is that there exist a subcollection of rows of Q_j whose sum is a vector having only odd entries.

(bII.2) m_j is even, $m_j = 2p_j$. In this case, $Ext^1(\mathbb{Z}_2, \mathbb{Z}_{m_j}) = \mathbb{Z}_2$, and Γ_j is either[19] the trivial extension $\mathbb{Z}_2 \times \mathbb{Z}_{m_j}$ or the nontrivial extension $\mathbb{Z}_{2m_j}$. We shall show that the first case is actually forbidden. For this, let us assume that $\Gamma_j = \mathbb{Z}_2 \times \mathbb{Z}_{m_j} = \mathbb{Z}_2 \times \mathbb{Z}_{2p_j}$. Remember from subsection 5.4.3 that the element s of $\mathbb{Z}_{m_j} = \mathbb{Z}_{2p_j}$ corresponds to an element $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_{n-2})$ of T^{n-2} which satisfies the first equations in (7.27) as well as the equation:

$$\prod_{\alpha=1}^{n-2} \lambda_\alpha^{q_j^{(\alpha)}} = e^{2\pi i \frac{s}{m_j}} . \tag{7.29}$$

Upon choosing $s = p_j = m_j/2$, we obtain an element $\boldsymbol{\lambda}^{(1)}$ of T^{n-2} which satisfies:

$$\begin{aligned} \prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(1)})^{q_k^{(\alpha)}} &= 1 \qquad \text{for } k \neq j \\ \prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(1)})^{q_j^{(\alpha)}} &= -1 \,. \end{aligned} \tag{7.30}$$

This corresponds to the element $(0, p_j)$ of $\mathbb{Z}_2 \times \mathbb{Z}_{m_j}$ On the other hand, the element $(1, 0)$ of $\mathbb{Z}_2 \times \mathbb{Z}_{m_j}$ corresponds to an element $\boldsymbol{\lambda}^{(2)} \in T^{n-2}$ which satisfies $\prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(2)})^{q_k^{(\alpha)}} = -1$ for $k \neq j$. Since $(1, 0)$ has order two in $\mathbb{Z}_2 \times \mathbb{Z}_{m_j}$, we have $(\boldsymbol{\lambda}^{(2)})^2 = ((\lambda_1^{(2)})^2, \ldots, (\lambda_{n-2}^{(2)})^2) = (1, \ldots, 1)$, so that $\lambda_\alpha^{(2)} = \pm 1$. Therefore, the product $\prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(2)})^{q_j^{(\alpha)}}$ equals $+1$ or -1. This

[19]If $m_j = 2p_j$ is even, then the groups $\mathbb{Z}_2 \times \mathbb{Z}_{m_j}$ and $\mathbb{Z}_{2m_j}$ are not isomorphic. To see why, notice that the first group has three order two elements, namely $(0, p_j), (1, 0)$ and $(1, p_j)$, while the second group has only one order two element (namely m_j).

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product cannot equal -1, since in that case $\boldsymbol{\lambda}^{(2)}$ would be a solution of the system (6.20) of subsection 6.6, thereby contradicting the assumption $\mathrm{U}(1)_{\rm eff} = \mathrm{U}(1)$. Therefore, we must have:

$$\prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(2)})^{q_k^{(\alpha)}} = -1 \qquad \text{for } k \neq j$$
$$\prod_{\alpha=1}^{n-2} (\lambda_\alpha^{(2)})^{q_j^{(\alpha)}} = 1\,. \tag{7.31}$$

Upon multiplying (7.30) and (7.31), we obtain an element $\boldsymbol{\lambda}^{(3)} = \boldsymbol{\lambda}^{(1)}\boldsymbol{\lambda}^{(2)} = (\lambda_1^{(1)}\lambda_1^{(2)}, \ldots, \lambda_{n-2}^{(1)}\lambda_{n-2}^{(2)})$ (corresponding to $(1, p_j) \in \mathbb{Z}_2 \times \mathbb{Z}_{m_j}$) which satisfies equations (6.20) of subsection 6.6, thereby contradicting the assumption that the $\mathbb{Z}_2$ subgroup of the projectivising $\mathrm{U}(1)$ acts nontrivially on X. This shows that Γ_j cannot be the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{m_j}$. Therefore, we must once again have $\Gamma_j = \mathbb{Z}_{2m_j}$. Up to an isomorphism, the extension (7.28) maps $u \in \mathbb{Z}_{m_j}$ into $2u \in \mathbb{Z}_{2m_j}$, so that $\mathbb{Z}_{m_j}$ corresponds to the subgroup $\{0, 2, 4, \ldots, 2(m_j - 1)\}$ of $\mathbb{Z}_{2m_j}$. We conclude that case (bII) always leads to $\Gamma_j = \mathbb{Z}_{2m_j}$. Combining everything, we obtain the following:

Proposition 2. *Let Γ_j demote the singularity group of Y along Y_j.*

(a) If $\mathrm{U}(1)_{\rm eff} = \mathrm{U}(1)/\mathbb{Z}_2$, then Γ_j is isomorphic with $\mathbb{Z}_{m_j}$.

(b) If $\mathrm{U}(1)_{\rm eff} = \mathrm{U}(1)$, then Γ_j coincides with $\mathbb{Z}_{m_j}$ or $\mathbb{Z}_{2m_j}$.

Upon combining with the results of appendix A, this shows that the integral Smith form of the matrix $\tilde{Q}_j$ (obtained by deleting the $j^{\rm th}$ and $(j+n)^{\rm th}$ rows of $\tilde{Q}$) is always of the type:

$$\tilde{Q}_j^{\rm ismith} = [\mathrm{diag}(1, \ldots, 1, t_j), 0]\,, \tag{7.32}$$

where t_j is either m_j or $2m_j$. To find which of the two possibilities arises in case (b), it suffices to determine t_j by computing this integral Smith form.

It is also easy to identify the action of Γ_j on the transverse quaternion coordinate $u_j = w_j^{(+)} + \mathbf{j} w_j^{(-)}$. If $\Gamma_j = \mathbb{Z}_{m_j}$, then we recover the transverse action on the locus $X_j \subset X$, which was determined in subsection 5.4.3. Therefore, the generator of $\mathbb{Z}_{m_j}$ acts as:

$$u_j \to e^{\frac{2\pi i}{m_j}} u_j \Leftrightarrow w_j^{(\pm)} \to e^{\pm\frac{2\pi i}{m_j}} w_j^{(\pm)}\,. \tag{7.33}$$

If $\Gamma_j = \mathbb{Z}_{2m_j}$, then $\mathbb{Z}_{m_j}$ is embedded as the subgroup $\{0, 2, 4, \ldots, 2(m_j - 1)\}$ of $\mathbb{Z}_{2m_j}$, acting as above. The generator of $\mathbb{Z}_{2m_j}$ does not belong to this subgroup, and corresponds to an element $\boldsymbol{\lambda} \in T^{n-2}$ which satisfies the second system in (7.27). Its square $\boldsymbol{\lambda}' = \boldsymbol{\lambda}^2 = (\lambda_1^2, \ldots, \lambda_{n-2}^2)$ corresponds to the generator of $\mathbb{Z}_{m_j}$ and satisfies the first system. Form subsection 5.4.3, we also know that $\boldsymbol{\lambda}'$ must satisfy:

$$\prod_{\alpha=1}^{n-2} (\lambda'_\alpha)^{q_j^{(\alpha)}} = e^{\frac{2\pi i}{m_j}}\,. \tag{7.34}$$

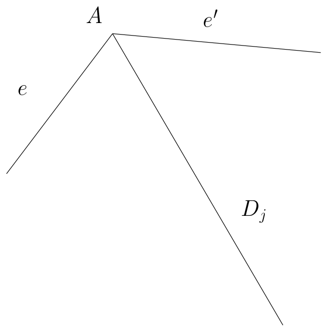

Figure 13: Vertices of Δ may lead to enhanced singularity types.

Therefore, we must have:

$$\prod_{\alpha=1}^{n-2} (\lambda_\alpha)^{q_j^{(\alpha)}} = e^{\frac{\pi i}{m_j}} . \tag{7.35}$$

This shows that the generator of $\mathbb{Z}_{2m_j}$ acts on the transverse coordinates as:

$$u_j \to e^{\frac{\pi i}{m_j}} u_j \Leftrightarrow w_j^{(\pm)} \to e^{\pm \frac{\pi i}{m_j}} w_j^{(\pm)} . \tag{7.36}$$

7.7 Singularities above the vertices of Δ

As mentioned above, the singularity type on the distinguished locus may be enhanced at the intersection points between the vertical and horizontal spheres. Each such point corresponds to a vertex of Δ. Consider a vertex A lying on the principal diagonal D_j supported by the line $h_j = \{a|a \cdot \nu_j = 0\}$. If e and e' are the edges of Δ meeting at A, then their sign vectors $\epsilon = \epsilon(e)$ and $\epsilon' = \epsilon(e')$ coincide except in position j:

$$\epsilon_k = \epsilon'_k \qquad \text{for } k \neq j\,, \qquad \epsilon_j = -\epsilon'_j\,. \tag{7.37}$$

Accordingly, the matrices Q_e and $Q_{e'}$ (defined as in subsection 7.5) coincide except for their j^{th} columns, which have opposite signs.

By swapping e and e', we can always assume that $\epsilon_j = +1$ and $\epsilon_j = -1$, and we shall do so in what follows. The spheres Y_e and $Y_{e'}$ are then defined by the constraints $w_k^{(-\epsilon_k)} = 0$ for $k \neq j$ and $w_j^{(-)} = 0$ (for Y_e) or $w_j^{(+)} = 0$ (for $Y_{e'}$). Their intersection point Y_A corresponds to $w_k^{(-\epsilon_k)} = 0$ for $k \neq j$ and $w_j^{(+)} = w_j^{(-)} = 0$. Accordingly, the T^{n-1} action on the nonvanishing coordinates $w_k^{(\epsilon_k)}$ $(k \neq j)$ is described by the $(n-1) \times (n-1)$ matrix:

$$\bar{Q}_A = \begin{bmatrix} \epsilon_1 \,\mathrm{col}(Q,1) & \dots & \epsilon_{j-1} \,\mathrm{col}(Q,j-1)\,, & \epsilon_{j+1} \,\mathrm{col}(Q,j+1) & \dots & \epsilon_n \,\mathrm{col}(Q,n) \\ 1 & \dots & 1 & 1 & \dots & 1 \end{bmatrix} . \tag{7.38}$$

The singularity type of Y at the point Y_A can now be extracted by computing the integral Smith form of $\bar{Q}_A$. We note that it is possible for Y_A to be a singular point even if Y_e, $Y_{e'}$ and Y_j are smooth in Y. In this case, Y_A is an isolated singular point, which induces a codimension *six* singularity of $\mathcal{C}(Y)$. Such singularities of the G_2 cone are expected to correspond to a conformal field theory in five dimensions, whose physics is rather poorly understood.

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7.8 Special isometries and good isometries

Recall from section 3 that a *special isometry* of the G_2 cone $\mathcal{C}(Y)$ is an isometry induced by one of the two-torus of triholomorphic isometries of X (via reduction to M, followed by a lift to $\mathcal{C}(Y)$). In this subsection, we are interested in characterizing those special isometries which fix the cones over Y_e or Y_j. Such isometries are relevant for understanding the IIA reduction of our models.

7.8.1 Special isometries which fix $\mathcal{C}(Y_e)$

Consider the U(1) subgroup of the two-torus of special isometries whose Lie algebra is given by $\mathbb{R}\nu_e \subset \mathbb{R}^2 = \mathrm{Lie}(T^2)$, where ν_e is the vector (7.22). The exact sequence (7.23) shows that this circle group corresponds to the specific cycle of the T^2 fibers of X_e which is killed by the projectivising U(1) quotient in order to produce the S^1 fiber of $Y_e \to e$. Therefore, this U(1) subgroup of T^2 is precisely the group of special isometries which fixes the cone $\mathcal{C}(Y_e)$.

7.8.2 Special isometries which fix $\mathcal{C}(Y_j)$

Remember that the T^2 fibers of $X \to \mathbb{R}^6$ (i.e. the orbits of the triholomorphic T^2 action on X) collapse to circles above $X_j - \{0\}$; thus a U(1) subgroup of T^2 fixes all points X. According to section 5, the Lie algebra of this U(1) is spanned by the toric hyperkähler generator ν_j. It is clear from section 7.3. that this is the subgroup which fixes every point of Y_j. Therefore, it also coincides with the special isometry subgroup which fixes the cone $\mathcal{C}(Y_j)$.

7.8.3 Good special isometries

When studying the type-IIA reduction of M-theory on $\mathcal{C}(Y)$, one is interested in 'good' special isometries of $\mathcal{C}(Y)$, i.e. special isometries which fix all of its singular loci. Let $E_{\rm sing}$ and $V_{\rm sing}$ be the collections of edges e of Δ and indices $j = 1, \ldots, n$ for which the associated loci Y_e or Y_j are singular in Y. According to the observations made above, 'good' special isometries of Y correspond to the subalgebra of $\mathbb{R}^2 = \mathrm{Lie}(T^2)$ given by the following intersection of one-dimensional spaces:

$$G = \left[\cap_{e \in E_{\rm sing}} (\mathbb{R}\nu_e)\right] \cap \left[\cap_{j \in V_{\rm sing}} (\mathbb{R}\nu_j)\right] . \tag{7.39}$$

It is clear that this intersection is generally zero, so that a good special isometry is usually impossible to find.

7.9 On the toric approach to finding the singularities of Y

We end this section with a short discussion of an alternate approach to finding the singularities of Y. This is based on a direct toric analysis starting from the embedding of Y in a toric variety, which was discussed in subsection 6.4. As we shall see, this somewhat naive method is rather inefficient, which is why we prefer the approach discussed above. The content of the present subsection is intended for readers familiar with toric geometry and is not needed for understanding the remainder of the paper.

If X is a good hyperkähler cone, then one can use an argument similar to that of appendix B to show that the affine variety $\mathcal{Z} \subset \mathbb{C}^{2n}$ is smooth outside the origin. In this case, all singularities of Y must be induced from singularities of the toric ambient space $\mathbb{T}$ discussed in subsection 6.4. It is well-known that a point z can be singular in $\mathbb{T}$ only if some of its homogeneous coordinates [47] $z_j = w_j^{(+)}, z_{j+n} = w_j^{(-)}$ vanishes. If $V(z) \subset \{1, \ldots, 2n\}$ is the set of indices j associated with the vanishing homogeneous coordinates of z, then the singularity type of $\mathbb{T}$ at z can be computed by considering the map $\mathbb{Z}^{r+1} \xrightarrow{\tilde{q}_N^*} \mathbb{Z}^N$ which is defined as the projection of $\tilde{q}^*$ of subsection 6.4. onto the sublattice $\mathbb{Z}^N$ of $\mathbb{Z}^{2n}$ associated with the complement N of V in the set $\{1, \ldots, 2n\}$. The associated matrix $\tilde{Q}_N$ is obtained from $\tilde{Q}$ by deleting all columns associated with the index set V. Computing the cokernel of $\tilde{q}_N^*$ gives a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^{r+1} \xrightarrow{\tilde{q}_N^*} \mathbb{Z}^N \xrightarrow{\tilde{g}_N} A_N \longrightarrow 0\,, \tag{7.40}$$

where the group A_N will generally contain torsion. In fact, it follows from appendix A that the orbifold group of $\mathbb{T}$ at z coincides with the torsion subgroup of A_N or with a $\mathbb{Z}_2$ quotient thereof (the second possibility arises since the T^{r+1} action on $\mathbb{C}^{2n}$ used to define $\mathbb{T}$ may fail to be effective). The former group can be determined by computing the integral Smith form of the matrix Q_N of q_N, which is obtained from $\tilde{Q}$ by keeping only those columns associated with the index set N, i.e. by deleting all columns associated with V.

Applying this procedure to the toric space $\mathbb{T}$ will typically give a large collection of singular loci. In general, only a small subset of these will intersect the twistor space; those which do not are irrelevant for our purpose. It should be clear from this observation that the direct toric approach is rather inefficient, since it involves a large number of singular loci of $\mathbb{T}$ which do not intersect Y; to find which of them do, one must check existence of solutions for a system of quadratic equations obtained by restricting (5.8) to each of these singular loci.

The procedure of the previous subsections avoids this problem by using information about the T^d fibration $Y \to \mathbb{R}^{3d}$ in order to implicitly solve the complex moment map constraints (5.8). Since in this paper we are mainly interested in G_2 cones, we have presented this procedure for the case $d = 2$ only; it is not hard to see that this approach generalizes to higher dimensions.

For the case $d = 2$, the analysis of the previous subsections tells us precisely which singular loci of $\mathbb{T}$ can intersect the twistor space:

(a) Singularities along loci given by simultaneous vanishing of $w_j^{(+)} = z_j$ and $w_j^{(-)} = z_{j+n}$ for some fixed j;

(b) Singularities along loci given by the simultaneous vanishing of precisely one coordinate $w^{(-\epsilon_j)}$ in *each* of the pairs $w_j^{(+)}, w_j^{(-)}$.

Loci of the first type intersect Y along Y_j, while loci of the second type intersect Y along Y_ϵ. According to our results, the second case can still lead to a void intersection; indeed, Y_ϵ is non-void if and only if the sign vector ϵ equals $\epsilon(e)$ for some edge e of Δ. Combining this with the previous discussion allows us to extract a 'simplified toric approach', which tests for singularities only along the distinguished locus. In fact, (a) and (b) tell us that it suffices to consider index sets V, N of the following forms:

1. $V = \{j, j+n\}$, $N = \{1, \ldots, 2n\} - V$, for the vertical loci Y_j

   and

2. $N = \{j + (1-\epsilon_j(e))\frac{n}{2} | j = 1, \ldots, n\}$, $V = \{1, \ldots, 2n\} - N$, for the horizontal loci Y_e.

Upon restricting to these loci, one obtains matrices $\tilde{Q}_N$ of the form $\tilde{Q}_j = \mathrm{delcols}(\tilde{Q}, j, j+n)$ (for Y_j) and $\bar{Q}_e = \begin{bmatrix} \epsilon_1(e)\,\mathrm{col}(Q,1) & \ldots & \epsilon_n(e)\,\mathrm{col}(Q,n) \\ 1 & \ldots & 1 \end{bmatrix}$. The singularity type along Y_e and Y_j results upon computing the integral Smith forms of these matrices. This approach still has the disadvantage that it is unclear why the singularity type is always a cyclic group, a fact which is easy to see in the alternate approach of the previous subsections. For the horizontal loci Y_e, a direct explanation of this fact in 'toric' language can be found in appendix D.

8. Examples

8.1 A model with three charges and a good isometry

Let $X = \mathbf{H}^5///_0\,\mathrm{U}(1)^3$, with (good and torsion-free) quaternion charge matrix:

$$Q = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}. \tag{8.1}$$

The toric hyperkähler generators ν_j ($j = 1..5$) are given by the columns of:

$$G = \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 3 & -3 & -1 & -2 \end{bmatrix}. \tag{8.2}$$

Since all generators are primitive, the hyperkähler cone X is smooth outside the apex. It is also clear from the form of Q that the projectivising $\mathrm{U}(1)$ acts effectively on this cone.

With the choice $\zeta = 1$, the vertices of Δ are given by the columns of the following matrix:

$$V = \begin{bmatrix} 0 & 1/3 & 2/5 & 3/8 & 1/5 & 0 & -1/3 & -2/5 & -3/8 & -1/5 \\ -1/9 & -2/9 & -1/5 & -1/8 & 0 & 1/9 & 2/9 & 1/5 & 1/8 & 0 \end{bmatrix}. \tag{8.3}$$

This is shown in figure 14 with the associated signs for its edges. The principal diagonals $D_j \subset h_j = \{a \cdot \nu_j = 0\}$ are given by the segments $D_1 = [1, 6]$, $D_2 = [4, 9]$, $D_3 = [2, 7]$, $D_4 = [5, 10]$, $D_5 = [3, 8]$.

One obtains two horizontal spheres of $\mathbb{Z}_3$ singularities associated with the opposite edges $e = [1, 2]$ and $-e = [7, 6]$ ($\nu_e = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$), and one vertical sphere[20] of $\mathbb{Z}_2$ singularities

[20] Since X has no singularities outside of its apex, we see that the singularity type along Y_j coincides with $\mathbb{Z}_{m_j} = \mathbb{Z}_1 = \{0\}$ for $j = 1, 3$ and with $\mathbb{Z}_{2m_j} = \mathbb{Z}_2$ for $j = 2$, where $m_1 = m_2 = m_3 = 1$ characterize the smooth loci $X_j \subset X$. This illustrates the two possibilities in case (b) of the Proposition of subsection 7.6.

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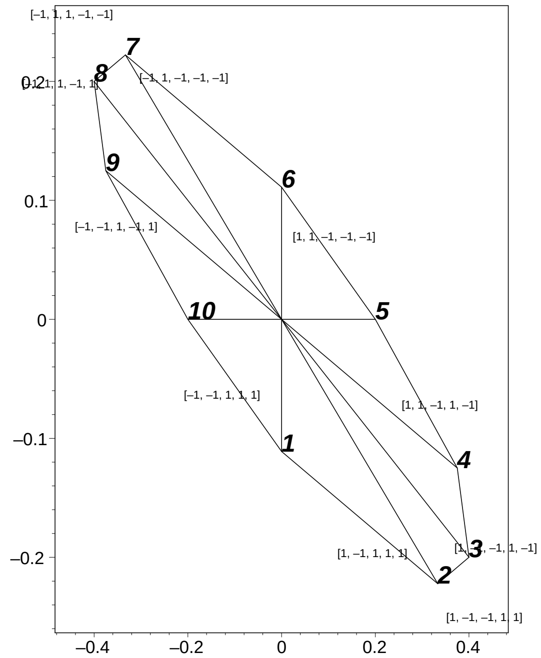

Figure 14: The polygon Δ.

from the principal diagonal D_2. The horizontal singularities are related by the antipodal map of the S^2 fibration $Y \to M$. One also obtains $\mathbb{Z}_9, \mathbb{Z}_6, \mathbb{Z}_7$ and $\mathbb{Z}_5$ singularities at the pairs of opposite vertices $\{1,6\}$, $\{2,7\}$, $\{3,8\}$ and $\{4,9\}$ respectively. Since $\nu_e = -3\nu_2 = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$, the G_2 cone admits a U(1) of good isometries with Lie algebra $\mathbb{R}\nu_2$.

8.2 An example with three charges and without a good isometry

Let us consider $X = \mathbf{H}^5 ///_0 \mathrm{U}(1)^3$, with:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 & 3 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 2 & 3 & 0 & -1 \\ 0 & 5 & 6 & 1 & -3 \end{bmatrix}. \tag{8.4}$$

Since the third column ν_3 of G fails to be primitive, the toric hyperkähler cone X has $\mathbb{Z}_3$ singularities along the locus $u_3 = 0$. The projectivising U(1) acts effectively on X.

The vertices of Δ (for $\zeta = 1$) are given by the columns of the matrix:

$$V = \begin{bmatrix} 0 & 2/5 & 5/11 & 3/8 & 1/7 & 0 & -2/5 & -5/11 & -3/8 & -1/7 \\ -1/15 & -1/5 & -2/11 & -1/8 & 0 & 1/15 & 1/5 & 2/11 & 1/8 & 0 \end{bmatrix}. \tag{8.5}$$

This is drawn in figure 15, together with the sign vectors of its edges. The principal diagonals $D_1, \ldots, D_5$ are given by the segments $[1,6], [3,8], [2,7], [5,10], [4,9]$, in this order.

One obtains a pair of horizontal spheres of $\mathbb{Z}_5$ singularities from the opposite edges $e = [1,2]$ and $-e = [6,7]$ (with $\nu_e = 5\nu_5 = \begin{bmatrix} -5 \\ -15 \end{bmatrix}$) as well as vertical spheres of $\mathbb{Z}_3$ and $\mathbb{Z}_2$ singularities[21] from the diagonals $D_3 = [2,7]$ and $D_5 = [4,9]$. One also has

[21] Notice that the singularity type of Y along Y_j coincides with that of X along X_j for $j \neq 5$, but it is 'doubled' (from $\mathbb{Z}_1 = \{0\}$ to $\mathbb{Z}_2$) for $j = 5$.

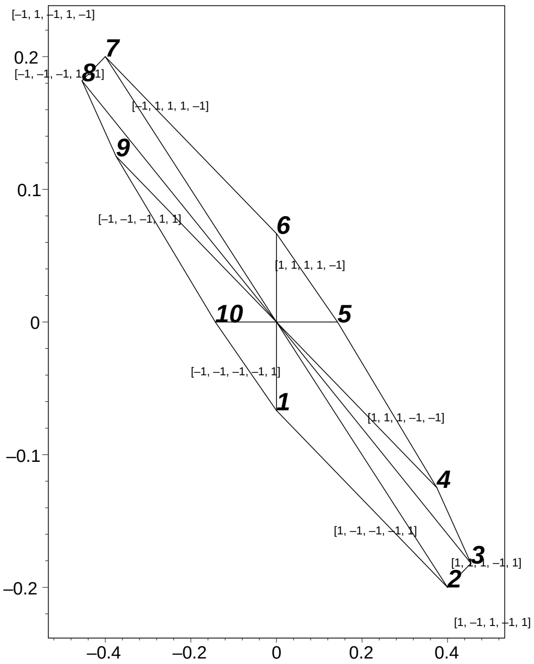

Figure 15: The polygon Δ.

$\mathbb{Z}_{15}, \mathbb{Z}_{11}, \mathbb{Z}_9, \mathbb{Z}_7$ and $\mathbb{Z}_6$ singularities at the points associated with the pairs of vertices $\{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}$ and $\{5,10\}$. The model has no good isometries.

8.3 A model with two families of singularities

For this example, we take $X = \mathbf{H}^4///_0\, \mathrm{U}(1)^2$, with:

$$Q = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & 1 & -2 \end{bmatrix}. \tag{8.6}$$

The toric hyperkähler cone is smooth outside its apex. For this model, the $\mathbb{Z}_2$ subgroup $\{-1, 1\}$ of the projectivising $\mathrm{U}(1)$ acts trivially on X. Indeed, the sum of the two rows of Q is a vector all of whose entries are odd. As explained above, this must be taken into account when computing the singularities of the twistor space.

With the choice $\zeta = 1$, the vertices of Δ are given by the columns of the matrix:

$$V = \begin{bmatrix} 0 & 1/2 & 1/2 & 1/4 & 0 & -1/2 & -1/2 & -1/4 \\ -1/6 & -1/3 & -1/4 & 0 & 1/6 & 1/3 & 1/4 & 0 \end{bmatrix}. \tag{8.7}$$

This polygon is shown in figure 16, together with the sign vectors of its edges. The principal diagonals $D_1, \ldots, D_4$ correspond to the segments $[1,5], [2,6], [4,8], [3,7]$.

One obtains two S^2's worth of $\mathbb{Z}_2$ singularities on the horizontal loci associated with the opposite edges $e = [3,4]$ and $-e = [7,8]$. The associated vector ν_e equals $\left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$. The model has no spheres of singularities on the vertical locus, but has isolated $\mathbb{Z}_3$ singularities at the points corresponding to the pairs of opposite vertices $\{1,5\}$ and $\{2,6\}$. This model has an S^1 of good isometries, with Lie algebra generated by the vector $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

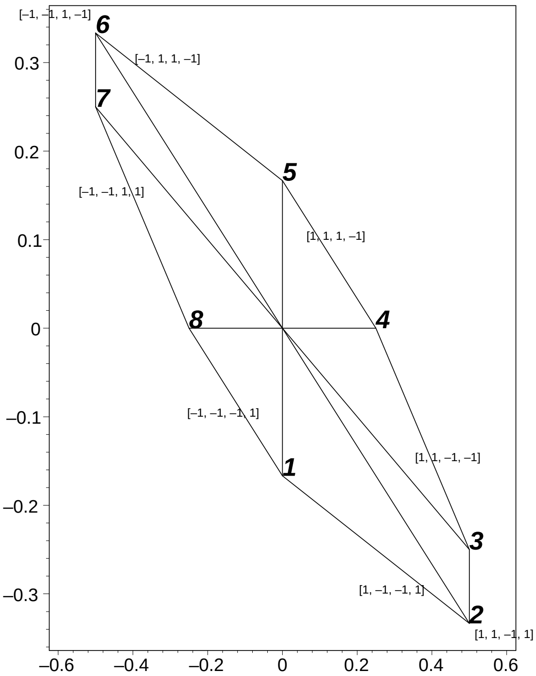

Figure 16: The polygon Δ.

9. Physical interpretation

The physical interpretation of our models is very similar to that of [50], and can be extracted by using similar arguments. Let us start with the abelian symmetries originating from the supergravity 3-form. Using the results of [28, 29], we find that $b_2(\mathcal{C}(Y)) = n - 1$, where n is the number appearing in the construction $X = \mathbf{H}^n///_0\, \mathrm{U}(1)^{n-2}$ of the associated hyperkähler cone. This gives a $\mathrm{U}(1)^{n-1}$ abelian symmetry produced by the reduction of the M-theory 3-form; as in [50], this symmetry is broken to $\mathrm{U}(1)^{n-2}$ for the deformed G_2 metric (3.5). Exactly as in [50], we can apply the argument of [14] to conclude that M-theory on our G_2 cones produces chiral fermions localized at the apex of the G_2 cone and charged under the abelian $\mathrm{U}(1)^{n-1}$ component of the resulting symmetry group.

Our models also produce nonabelian symmetries originating from codimension four singularities of the G_2 metric. As is well-known (see [53] and references therein) quantum field theories can be geometrically engineered using local description of singularities of type-IIA compactification spaces. As discussed in [54], the same idea extends to M-theory compactified on singular spaces of G_2 holonomy. In the paper cited, it was argued that $A - D - E$ singularities in codimension four lead to an $N = 1$ super Yang-Mills theory carrying corresponding $A - D - E$ gauge group and localized at the singularity. As explained in section 7 and illustrated in section 8, codimension four singularities in our models are always cyclic, and therefore of type A_{n-1}. This gives an $\mathrm{SU}(n)$ gauge theory living at every codimension four singular locus with $\mathbb{Z}_n$ singularity type. The location of such singularities and the their singularity type can be extracted with the methods of section 7.

Codimension six singularities of the G_2 metric, if present, are less well understood. They correspond to isolated singular points of the twistor space, and are presumably related to certain conformal field theories in five dimensions. We have encountered such singularities in the examples of section 8.

Since our G_2 spaces have a two-torus of isometries, we can choose a $\mathrm{U}(1) \subset T^2$ and dimensionally reduce M-theory to IIA. The resulting type-IIA background contains a non-vanishing RR one-form and hence will correspond to a configuration of D6-branes. As in [50, 52, 55] the D-brane worldvolumes are comprised of $\mathbb{R}^{3,1}$, the radial direction r and a singular locus Y_e or Y_j in Y, which is preserved (as a set) by the U(1) isometry. From subsection 7.8.3, we know that good isometries (namely isometries which fix all points of every singular locus) are very rare. If such isometries do not exist, then the IIA background is strongly coupled along those six-branes associated with singular loci which are not pointwise invariant under the isometry chosen for the reduction. If a good isometry exists, then reducing through it leads to a IIA solution which describes a weakly coupled system of D6-branes. In fact we did find such models in the examples of subsections 8.1 and 8.3 (and many more such models can be constructed upon using our criteria). For the example of subsection 8.1., the vector ν_e associated with the edge $e = [1, 2]$ was equal to $-3\nu_2$. Reducing along the isometry generated by ν_2 we obtain weakly-coupled D6-branes associated with both the horizontal and the vertical loci. In the example of subsection 8.3, one has a good isometry generated by the vector ν_e with $e = [3, 4]$, since there are no singularities on the vertical locus.

After reduction to IIA, the residual U(1) isometry of the background can be used to obtain a T-dual IIB description. From the discussion of section 5, we know that the T^2 fibers describing the triholomorphic orbits in the hyperkähler cone X are collapsed to circles on the loci X_j. When reducing X_j to Y_j through the projectivising U(1) action, these S^1 fibers become circles along the sphere Y_j (namely the circle fibers of the fibration of Y_j over the principal diagonal D_j of Δ.) Therefore, the T-duality mapping our system to IIB acts along a worldvolume direction of the vertical branes. This converts a von Neumann direction into a Dirichlet direction, thereby leading to a IIB fivebrane. Since we are dualizing a von Neumann into a Dirichlet direction, we expect that T-dualization of the vertical 6-branes will produce *delocalized* 5-branes in the IIB background; the delocalization should occur along the T-dual of the S^1 fiber of $Y_j \to D_j$. A similar argument applies to those IIA 6-branes which are associated with horizontal loci Y_e. In this case, we know from section 5 that the triholomorphic T^2 orbits of X are not collapsed on the loci X_e. However, the projectivising U(1) now acts along the T^2 fibers of X_e, which therefore descend to circles upon reduction to Y_e; these are the S^1 fibers of $Y_e \to e$. Once again, we find that T-dualization is performed along a worldvolume direction, and therefore vertical IIA 6-branes will correspond to delocalized five-branes in IIB. If the IIA reduction is performed through a good isometry, then the weak coupling arguments given above show that the IIA solution describes a system of D6-branes, while the T-dual IIB background describes a system of delocalized D5-branes. In this case, the IIA description provides an alternate explanation of the origin of chiral fermions. If the isometry used in the reduction is not good (which is necessarily the case for models which do not admit a good isometry) then we obtain a set of strongly coupled 'branes' in IIA and we expect a set of strongly coupled dual loci in IIB. The physical interpretation of these loci is less clear, though the associated supergravity solutions can be analyzed explicitly [56]. However, it is natural to extend our conclusions to these cases, and propose that the resulting M-theory backgrounds lead to nonabelian gauge theories whose gauge groups are constructed in the manner discussed above.

10. Conclusions

By using the relation between an Einstein self-dual orbifold M, its twistor space Y and its hyperkähler cone X, we developed methods to identify the location and type of singularities of the twistor space of a compact, Einstein self-dual orbifold of positive scalar curvature which admits a two-torus of isometries. This amounts to considering the class of models for which the hyperkähler cone X of M is 'toric hyperkähler' in the sense of [34], i.e. admits a presentation as a toral hyperkähler quotient $X = \mathbf{H}^n///_0 \,\mathrm{U}(1)^{n-2}$. This is precisely the class of ESD spaces for which one expects a simplification in view of the work of [39]. Upon combining our methods with the construction of [37, 38], we obtained an algorithm for analyzing the associated G_2 cones $\mathcal{C}(M)$, which allows us to extract the low energy gauge group produced by M-theory on such backgrounds. This identifies the basic physics of such models, which form a vast generalization of those considered in [50]. Since the G_2 cones belonging to this family admit a T^2 of isometries, such M-theory backgrounds have a two-torus of T-dual type-IIA and IIB descriptions, whose physics describes systems of strongly and/or weakly coupled 6-branes, T-dual to systems of delocalized fivebranes. By using abstract geometric arguments, we showed that the generic model in this family does not allow for a description in terms of weakly coupled D-branes only, which means that its type-II reduction is not amenable to conformal field theory techniques. Those (non-generic) models which *do* admit such a description can be identified by a simple criterion, and we presented some examples of this type. The techniques developed in this paper are computationally quite effective and they allow for an explicit analysis of any representative of this large family of models (the complexity of the computations involved depends on the number n).

Since the ESD spaces of interest admit a two-torus of isometries, they belong to the class recently considered in the work of Calderbank and Pedersen, who wrote down the explicit ESD metric for the most general space of this type. When combined with the construction of [37, 38], this allows for an explicit description of the associated G_2 metrics, and therefore of the IIA backgrounds obtained by performing the KK reduction, together with their type-IIB duals. This analysis provides an independent confirmation of the conclusions of the present paper, as we will show in [56].

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A. On maps of lattices and associated maps of tori

This appendix collects a few basic facts about maps of lattices. Some of the properties described below are well-known and used intensively, for example, in toric geometry [42, 44, 45, 47, 46]. By definition, a lattice is a free, finitely generated abelian group, which is the same as a free, finitely generated $\mathbb{Z}$-module. A map of lattices is a group morphism between such objects, which is the same as a morphism of $\mathbb{Z}$-modules. We remind the reader that the ring $\mathbb{Z}$ of integers is a PID (principal ideal domain),[22] which is why the homological algebra of $\mathbb{Z}$-modules has a particularly simple form [30]. In particular, a finitely generated $\mathbb{Z}$-module is projective if and only if it is free, which is equivalent with it being torsion free.

A.1 The structure theorem of lattice maps

The following result is well-known:

Theorem 1. *Given a lattice map $f : \mathbb{Z}^r \to \mathbb{Z}^n$, one can always find integral bases $v_1, \dots, v_r$ of $\mathbb{Z}^r$ and $u_1, \dots, u_n$ of $\mathbb{Z}^n$ such that $f(v_\alpha) = t_\alpha u_\alpha$ for all $\alpha = 1, \dots, r$, where t_α are non-negative integers satisfying the division relations $t_1 | t_2 | \dots | t_r$.*

This is simply the structure theorem of finitely-generated abelian groups, adapted to our context. If e'_α and e_j are the canonical bases of $\mathbb{Z}^r$ and $\mathbb{Z}^n$, then f can be described by an $n \times r$ matrix $F_{j\alpha} = f_j^{(\alpha)}$, whose entries are defined through:

$$f(e'_\alpha) = \sum_{j=1}^{n} f_j^{(\alpha)} e_j \, . \tag{A.1}$$

The theorem says that there exist matrices $U \in GL(n, \mathbb{Z})$ and $V \in GL(r, \mathbb{Z})$ such that $U^{-1}FV$ is in *integral Smith form*:

$$U^{-1}FV = F^{\mathrm{ismith}} = \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \end{bmatrix} . \tag{A.2}$$

Indeed, V is the matrix whose columns are v_α and U is the matrix whose columns are u_j. We shall mostly be interested in the case $r \leq n$. Then the product $\prod_{j=\alpha}^{r} t_\alpha$ equals the so-called r^{th} *discriminantal divisor* $\mathfrak{g}(F)$ of the matrix F, i.e. the greatest common divisor of all $r \times r$ minor determinants of F.

The integers $t_1, \dots, t_r$ are independent of the choice of bases v_α, u_j with properties as in the theorem, and are called the *torsion coefficients* of the lattice map f (also known

[22] Recall that a PID is a commutative ring without divisors of zero, for which every ideal is principal, i.e. generated by a single element.

as the *invariant factors* of the associated matrix F). They coincide with the torsion coefficients of the torsion subgroup of the cokernel of f, i.e. one has a group isomorphism $\mathbb{Z}^n/f(\mathbb{Z}^r) = \mathbb{Z}_{t_1} \oplus \cdots \oplus \mathbb{Z}_{t_r} \oplus \mathbb{Z}^{n-r}$, where $\mathbb{Z}_t := \mathbb{Z}/t\mathbb{Z}$ and $\mathbb{Z}^{n-r}$ is defined to be zero if $n \leq r$. If $t = 0$, then $\mathbb{Z}_0$ is a copy of $\mathbb{Z}$. If $t = 1$, then $\mathbb{Z}_1 = \mathbb{Z}/\mathbb{Z} = 0$. It is clear that the map f is injective if and only if $t_r > 0$, in which case all torsion coefficients are non-vanishing.

The following is an immediate consequence of the structure theorem:

Proposition 3. *Let $f : \mathbb{Z}^r \to \mathbb{Z}^n$ be an* injective *map of lattices (thus $r \leq n$ and $f(\mathbb{Z}^r)$ is a sublattice of $\mathbb{Z}^n$). Then the following are equivalent:*

(a) $\mathrm{coker}(f) = \mathbb{Z}^n/f(\mathbb{Z}^r)$ *is torsion free (and thus free, i.e. a copy of the lattice $\mathbb{Z}^d$, where $d = n - r$)*

(b) $f(\mathbb{Z}^r)$ is a direct summand in $\mathbb{Z}^n$

(c) All torsion coefficients of f equal one

(d) The r^{th} discriminantal divisor of the matrix F equals one.

In this case, one has a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{f} \mathbb{Z}^n \xrightarrow{g} \mathbb{Z}^d \longrightarrow 0\,. \tag{A.3}$$

Another basic result is:

Proposition 4. *Let $g : \mathbb{Z}^n \to \mathbb{Z}^d$ be a map of lattices. Then the kernel of g is a lattice.*

Proof. The ring of integers is a PID, so that all projective $\mathbb{Z}$-modules are free, and any submodule of a projective module is free [30].

Observation Note that any short exact sequence of lattices (A.3) is split exact. Indeed, $Ext^1(\mathbb{Z}^r, \mathbb{Z}^d)$ vanishes since $\mathbb{Z}^r$ is a projective module. In particular, the dual sequence

$$0 \longrightarrow (\mathbb{Z}^d)^* \xrightarrow{g^*} (\mathbb{Z}^n)^* \xrightarrow{f^*} (\mathbb{Z}^r)^* \longrightarrow 0\,, \tag{A.4}$$

where $(\mathbb{Z}^n)^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$ etc. is also exact.

A.2 Some properties of torus maps

A map of lattices $f : \mathbb{Z}^r \to \mathbb{Z}^n$ induces a map of tori $f : T^r \to T^n$ upon tensoring with the abelian group U(1). If f is injective, then the short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{f} \mathbb{Z}^n \xrightarrow{g} A \longrightarrow 0 \tag{A.5}$$

(where $A = \mathrm{coker}(f)$) induces a long exact Tor sequence [30] which collapses to four terms since $\mathbb{Z}^n$ is flat so that $\mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z}^n, \mathrm{U}(1))$ vanishes:

$$0 \longrightarrow \Gamma \longrightarrow T^r \xrightarrow{f} T^n \xrightarrow{g} A \otimes_{\mathbb{Z}} \mathrm{U}(1) \longrightarrow 0\,, \tag{A.6}$$

where $\Gamma = \mathrm{Tor}^{\mathbb{Z}}(A, \mathrm{U}(1))$. The sequence (A.6) collapses to a short exact sequence if and only if A is torsion-free; by the result above, this happens precisely when f has trivial torsion coefficients, in which case one has $A = \mathbb{Z}^d$ and $A \otimes_{\mathbb{Z}} \mathrm{U}(1) = T^d$. Otherwise, (A.6) shows that the induced map of tori $f : T^r \to T^n$ will fail to be injective; its (finite) kernel Γ is the torsion group $\mathrm{Tor}^{\mathbb{Z}}(A, \mathrm{U}(1))$. Let $t_1, \ldots, t_r$ be the torsion coefficients of f and v_α, u_j be bases of $\mathbb{Z}^r$ and $\mathbb{Z}^n$ such that $f(v_\alpha) = t_\alpha u_\alpha$. We let $T(A)$ denote the torsion subgroup of A.

Proposition 5. *Assume that f is injective. Then Γ and the torsion subgroup $T(A)$ are both isomorphic with $\mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_r}$. An isomorphism $\psi : \mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_r} \to \Gamma$ is given by:*

$$\psi(s_1 \ldots s_r) = (e^{2\pi i \sum_{\alpha=1}^r v_\alpha^{(\beta)} \frac{s_\alpha}{t_\alpha}})_{\beta=1,\ldots,r}\,, \tag{A.7}$$

where $v_\alpha = \sum_{\beta=1}^r v_\alpha^{(\beta)} e'_\beta$, with (e'_β) the canonical basis of $\mathbb{Z}^r$. Here Γ is viewed as a subgroup of $T^r = \mathrm{U}(1)^r$, embedded by inclusion.

Proof. Writing $f(e'_\alpha) = \sum_{j=1}^n f_j^{(\alpha)} e_j$, the embedding $T^r \to T^n$ is given by:

$$(\lambda_\alpha)_{\alpha=1,\ldots,r} \to \left(\prod_{\alpha=1}^r \lambda_\alpha^{f_j^{(\alpha)}}\right)_{j=1,\ldots,n}\,, \tag{A.8}$$

where $\lambda_\alpha \in \mathrm{U}(1)$. The group Γ consists of the solutions $\lambda \in T^r$ of the system:

$$\prod_{\alpha=1}^r \lambda_\alpha^{f_j^{(\alpha)}} = 1 \qquad \text{for all } j = 1, \ldots, n\,. \tag{A.9}$$

Since f is injective, the solution set is finite. Writing $\lambda_\beta = e^{2\pi i \phi_\beta}$ (with $\phi_\beta \in \mathbb{R}/\mathbb{Z}$) reduces this system to:

$$f(\phi) = 0 \qquad \text{in } (\mathbb{R}/\mathbb{Z})^n\,. \tag{A.10}$$

It is clear that all solutions $\phi = (\phi_1, \ldots, \phi_r)$ belong to $(\mathbb{Q}/\mathbb{Z})^r$. With the structure given by addition, these form a group isomorphic with Γ. Using the structure theorem for f, we write $\phi = \sum_{\alpha=1}^r \phi^{(\alpha)} v_\alpha$, which reduces (A.10) to the form:

$$t_\alpha \phi^{(\alpha)} = 0 \qquad \text{in } \mathbb{R}/\mathbb{Z} \text{ for all } \alpha\,. \tag{A.11}$$

The general solution is $\phi^{(\alpha)} = s_\alpha / t_\alpha$, with $s_\alpha \in \mathbb{Z}_{t_\alpha}$ (this is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$). Since $\phi_\beta = \sum_\alpha \phi^{(\alpha)} v_\alpha^{(\beta)}$, we find that Γ coincides with $\mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_r}$ via the isomorphism ψ. The remaining statement follows from $T(A) = T(\mathbb{Z}^n / f(\mathbb{Z}^r)) = \langle u_1, \ldots, u_r \rangle_{\mathbb{Z}} / \langle t_1 u_1, \ldots, t_r u_r \rangle_{\mathbb{Z}} = \mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_r}$.

Consider the short exact sequence of lattices (A.3) and a partition of $\{1, \ldots, n\}$ into subsets V and N, where V has at most d (and hence N has at least r) elements. Associated to this partition is a direct sum decomposition $\mathbb{Z}^n = \mathbb{Z}^V \oplus \mathbb{Z}^N$, with $\mathbb{Z}^V = \oplus_{j \in V} \mathbb{Z} e_j$ and $\mathbb{Z}^N = \oplus_{j \in N} \mathbb{Z} e_j$, where (e_j) is the canonical basis of $\mathbb{Z}^n$. We let j_V and j_N denote the

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z}^V \arrow[r, "id"] \arrow[d, "j_V"] & \mathbb{Z}^V \arrow[d, "g_V"] & \\
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "f"] \arrow[d, "id"] & \mathbb{Z}^n \arrow[r, "g"] \arrow[d, "p_N"] & \mathbb{Z}^d \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathbb{Z}^r \arrow[r, "f_N"] & \mathbb{Z}^N \arrow[r, "\beta"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Figure 17: Exact sequences.

injections of $\mathbb{Z}^V$ and $\mathbb{Z}^N$ into $\mathbb{Z}^n$ and p_V, p_N the projections of $\mathbb{Z}^n$ onto these sublattices. We have a split short exact sequence:

$$0 \longrightarrow \mathbb{Z}^V \xrightarrow{j_V} \mathbb{Z}^n \xrightarrow{p_N} \mathbb{Z}^N \longrightarrow 0\,, \tag{A.12}$$

with left and right splittings given by p_V and j_N:

$$p_V \circ j_V = id_{\mathbb{Z}^V}\,, \qquad p_N \circ j_N = id_{\mathbb{Z}^N}\,. \tag{A.13}$$

Let us define $f_N := p_N \circ f$ and $g_V := g \circ j_V$. Computing the cokernel of f_N gives an exact sequence:

$$\mathbb{Z}^r \xrightarrow{f_N} \mathbb{Z}^N \xrightarrow{\beta} A \longrightarrow 0\,, \tag{A.14}$$

where the group $A = \mathbb{Z}^N/im(f_N)$ may have torsion, even though the cokernel of f is torsion-free. The situation is summarized in figure 17.

Proposition 6. *The following are equivalent:*

(a) The map f_N is injective

(b) The map g_V is injective

In this case, there exists a unique morphism α from $\mathbb{Z}^d$ to A which closes figure 17 to a commutative diagram. Moreover, the resulting vertical sequence:

$$0 \longrightarrow \mathbb{Z}^V \xrightarrow{g_V} \mathbb{Z}^d \xrightarrow{\alpha} A \longrightarrow 0 \tag{A.15}$$

is exact. This allows us to compute $A = \mathrm{coker}(f_N)$ as the cokernel of g_V.

Proof. Assume that f_N is injective. If x belongs to $\ker g_V$, then $g(j_V(x)) = g_V(x) = 0$ so $j_V(x) \in \ker g = imf$, which gives an element $y \in \mathbb{Z}^r$ such that $j_V(x) = f(y)$. Now $f_N(y) = p_N(f(y)) = p_N(j_V(x)) = 0$ since $p_N \circ j_V$ vanishes. Hence $f_N(y) = 0$ which implies $y = 0$ by injectivity of f_N. Thus $j_V(x) = 0$ and so $x = 0$ since j_V is injective. Therefore $\ker g_V = 0$ and g_V is injective.

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To prove the converse implication, it suffices to notice that the diagram in figure 17 is symmetric with respect to the northwest-southeast diagonal; hence the proof of $(b) \Rightarrow (a)$ is formally identical to that of $(a) \Rightarrow (b)$.

Let us now assume that (a) (and thus (b)) hold. Then existence and uniqueness of α follows by applying the 3-lemma to the two horizontal exact sequences. Surjectivity of α follows trivially from surjectivity of β and p_N and commutativity of the lower right square.

It remains to prove exactness of (A.15) at the middle. For this, consider an element x in the kernel of α. Then $x = g(y)$ for some y in $\mathbb{Z}^n$, and $\beta(p_N(y)) = \alpha(g(y)) = \alpha(x) = 0$, which shows that $p_N(y)$ lies in $\ker\beta = imf_N$. Thus $p_N(y) = f_N(z)$ for some z in $\mathbb{Z}^r$. Since $f_N = p_N \circ f$, this gives $y = f(z) + t$, with $t \in \ker p_N = imj_V$, so that $y = f(z) + j_V(s)$ for some s in $\mathbb{Z}^V$. Hence $x = g(y) = g(j_V(s)) = g_V(s)$, where we used exactness of the upper horizontal sequence as well as commutativity of the upper right square. It follows that $\ker\alpha \subset img_V$. The opposite inclusion follows from $\alpha \circ g_V = \alpha \circ g \circ j_V = \beta \circ p_N \circ j_V = 0$, since $p_N \circ j_V = 0$.

Observation. Assume that the hypothesis (a) (and thus (b)) of Proposition 6 holds. Upon tensoring the diagram of figure 17 with U(1), we obtain the diagram shown in figure 18, where the groups Γ and Γ' are isomorphic. Computing Γ from the bottom horizontal sequence, one obtains a presentation of type (A.7), where this time $v_\alpha^{(N)}$ is a basis of $\mathbb{Z}^r$ such that $f_N(v_\alpha^{(N)}) = t_\alpha^{(N)} u_\alpha^{(N)}$ for some basis $u_j^{(N)}$ of $\mathbb{Z}^N$. Notice that Γ maps to T^V through the composite $p_V \circ f$. Upon using expression (A.7), we find that this map is given by:

$$\lambda_j = \prod_{\alpha=1}^{r} e^{\frac{2\pi i}{t_\alpha^{(N)}} s_\alpha \sum_{\beta=1}^{r} f_j^{(\beta)} v_\alpha^{(N,\beta)}} \qquad (j \in V)\,, \tag{A.16}$$

so that the generator of $\mathbb{Z}_{t_\alpha^{(N)}}$ embeds into T^V as $(e^{\frac{2\pi i}{t_\alpha^{(N)}} \sum_{\beta=1}^{r} f_j^{(\beta)} v_\alpha^{(N,\beta)}})_{j\in V}$. In matrix language, we have $F_N V_N = U_N F_N^{\mathrm{ismith}}$, where V_N, U_N are invertible $r \times r$ and $|N| \times |N|$ integral matrices whose columns are $v_\alpha^{(N)}$ and $u_j^{(N)}$ ($|N|$ denotes the cardinality of the set N, and F_N is the $|N| \times r$ matrix of the map f_N, which results from the matrix F of f by deleting the rows associated with the index set V). Then (A.16) says that the generator of $\mathbb{Z}_{t_\alpha^{(N)}}$ embeds into T^V according to the element $(FV_N)_{j\alpha}$ of the matrix FV_N (where $j \in V$). The collection of such elements forms the matrix obtained from FV_N by deleting the rows associated with the index set N.

If one uses the vertical rightmost sequence instead, then one determines Γ from the invariant factors t'_k of the matrix G_V. In this case, Γ embeds into T^V by the uppermost vertical arrow, which according to (A.7) is given by:

$$\lambda_j = \prod_{k\in V} e^{\frac{2\pi i}{t'_k} s_k v_k^{'(j)}} \qquad (j \in V)\,, \tag{A.17}$$

where this time v'_k, u'_j are bases of $\mathbb{Z}^V$ and $\mathbb{Z}^d$ such that $g_V(v'_k) = t'_k u'_k$. These are the columns of invertible integral matrices U', V' which bring the matrix G_V (obtained from G by deleting the columns associated with N) to its integral Smith form.

$$
\begin{tikzcd}
& & & & 1 \arrow[d] & \\
& & & 1 \arrow[d] & \Gamma' \arrow[d] & \\
& & & T^V \arrow[r, "id"] \arrow[d, "j_V"] & T^V \arrow[d, "g_V"] & \\
& 1 \arrow[r] & T^r \arrow[r, "f"] \arrow[d, "id"] & T^n \arrow[r, "g"] \arrow[d, "p_N"'] & T^d \arrow[r] \arrow[d, "\alpha"] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] & T^r \arrow[r, "f_N"] & T^N \arrow[r, "\beta"] \arrow[d] & A \otimes_{\mathbb{Z}} U(1) \arrow[r] \arrow[d] & 1 \\
& & & 1 & 1 &
\end{tikzcd}
$$

Figure 18: The associated exact sequences of tori.

From figure 18, we also obtain:

$$j_V(\Gamma') = j_V(\ker g_V) = \ker g \cap imj_v = imf \cap \ker p_N = f(\ker f_N) = f(\Gamma)\,. \qquad \text{(A.18)}$$

Since j_V and f are injective maps of tori, this allows us to determine one of the isomorphic presentations Γ, Γ' given the other. The meaning of this is as follows. The group Γ is defined as the kernel of the map $f_N : T^r \to T^N$. Proposition 6 shows that this group is isomorphic with $\Gamma' := \ker(g_V : T^V \to T^d)$, while relation (A.18) shows that $\Gamma = f^{-1}(j_V(\Gamma'))$. In many applications, it is easier to determine Γ' and reconstruct its embedding Γ into T^r by using the later relation.

A particular case. If the set V has only one element, say $V = \{j\}$, then $g_V(\mathbb{Z}^V) = \mathbb{Z}\nu_j$ (where $\nu_j = g(e'_j)$ is the j^{th} column of the matrix G of g) and $A = \mathbb{Z}^d/\mathbb{Z}\nu_j$. If $m_j = gcd(\nu_j^1, \ldots, \nu_j^d)$, then we can write $\nu_j = m_j w_1$, where w_1 is a primitive integral vector in $\mathbb{Z}^d$. This vector can be completed to an integral basis $w_1, \ldots, w_d$ of $\mathbb{Z}^d$, which gives $A \approx \mathbb{Z}_{m_j} \times \mathbb{Z}^{d-1}$; in particular, G_V has a single invariant factor, given by $t'_1 = m$. Hence $\Gamma = T(A) = \mathbb{Z}_{m_j}$. In this case, the matrix G_V reduces to the column vector ν_j and there is a single basis vector $v_1 = e_j$ for the lattice $\mathbb{Z}^V = \mathbb{Z}e_j \approx \mathbb{Z}$. The embedding of $\mathbb{Z}_{m_j}$ into $T_V = \mathrm{U}(1)$ takes the generator of $\mathbb{Z}_{m_j}$ into $e^{2\pi i/m_j}$, while the embedding of $\mathbb{Z}_{m_j}$ into T^n effected by j_V takes this generator into the element $\Lambda = (1, \ldots, 1, e^{2\pi i/m_j}, 1, \ldots, 1) \in T^n$, where $e^{2\pi i/m_j}$ sits in position j. If we let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathrm{U}(1)^r$ parameterize elements of T^r, then relation (A.18) tells us that Γ is the multiplicative group of solutions to the system of equations:

$$\begin{aligned}
\prod_{\alpha=1}^{r} \lambda_\alpha^{f_k^{(\alpha)}} &= 1 \qquad \text{for } k \neq j \\
\prod_{\alpha=1}^{r} \lambda_\alpha^{f_j^{(\alpha)}} &= e^{\frac{2\pi i s}{m_j}}\,, \qquad s \in \mathbb{Z}_{m_j}\,.
\end{aligned} \qquad \text{(A.19)}$$

Since $f : T^r \to T^n$ is an injective map of tori, this makes it obvious why the group of such solutions is isomorphic with $\mathbb{Z}_{m_j}$, and gives the embedding of the latter group into

T^r. Note that Γ is defined by the first $n-1$ equations of (A.19), which are associated with the non-injective map $f_N : T^r \to T^N$. Knowledge of the last equation (which follows automatically from the first under our assumptions) supplements these $n-1$ conditions by the last constraint in (A.19). That is, any solution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-2})$ of the first $n-1$ equations will automatically satisfy the last equation of (A.19) for some uniquely determined element $s = s_{\boldsymbol{\lambda}} \in \mathbb{Z}_{m_j}$, and the map $\boldsymbol{\lambda} \to s_{\boldsymbol{\lambda}}$ gives the isomorphism $\Gamma \approx \mathbb{Z}_{m_j}$. This allows us to determine the structure of Γ without explicitly solving the original $n-1$ equations.

B. Singularities of toric hyperkähler cones

B.1 Good toric hyperkähler cones

Consider a toric hyperkähler cone $X = \mathbf{H}^n///_0 T^r$, defined by a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{q^*} \mathbb{Z}^n \xrightarrow{g} \mathbb{Z}^d \longrightarrow 0\,, \tag{B.1}$$

where q^* is the transpose of a map $q : \mathbb{Z}^n \to \mathbb{Z}^r$. Let Q, G be the matrices of q and g with respect to the canonical bases of the appropriate lattices. Let $\vec{\mu} : \mathbf{H}^n \to \mathbb{R}^r \otimes \mathbb{R}^3$ be the hyperkähler moment map of the associated T^r action and $\mathcal{N} = \vec{\mu}^{-1}(0) \subset \mathbf{H}^n$ be its zero level set. We recall the following:

Fact. The image of the differential $d_u\vec{\mu} : \mathbf{H}^n \to \mathbb{R}^r \otimes \mathbb{R}^3$ at a point $u \in \mathcal{N}$ coincides with $stab_{T^r}(u)^\perp \otimes \mathbb{R}^3$, where $stab_{T^r}(u) \subset \mathbb{R}^r$ is the Lie algebra of the stabilizer $Stab_{T^r}(u)$ of u in T^r.

It follows that the point u is smooth in $\mathcal{N}$ if and only if $stab_{T^r}(u)$ vanishes, i.e. $Stab_{T^r}(u)$ is a finite group. Given a point $u \in \mathcal{N}$, we define $V(u) = \{j \in \{1, \ldots, n\} | u_j = 0\}$ and $N(u) = \{j \in \{1, \ldots, n\} | u_j \neq 0\}$. This gives a partition $\{1, \ldots, n\} = V(u) \cup N(u)$ and associated decompositions $\mathbb{R}^n = \mathbb{R}^V \oplus \mathbb{R}^N$, $\mathbb{Z}^n = \mathbb{Z}^V \oplus \mathbb{Z}^N$ and $q^* = q^*_V \oplus q^*_N$. Here $q^*_V = p_V \circ q^*$ and $q^*_N = p_N \circ q^*$, where p_N, p_V are the projections of $\mathbb{Z}^n$ onto $\mathbb{Z}^N$ and $\mathbb{Z}^V$. It is clear that $stab_{T^n}(u)$ (the Lie algebra of the stabilizer of u in the 'diagonal' torus T^n) equals $\mathbb{R}^V$ (indeed, only the vanishing coordinates of u are invariant under the diagonal T^n action $u_j \to \Lambda_j u_j$ on $\mathbf{H}^n$). Therefore, $stab_{T^r}(u)$ equals the preimage of the subspace $\mathbb{R}^V \subset \mathbb{R}^n$ through (the real extension of) the map q^*, i.e. the kernel of the map $q^*_N : \mathbb{R}^r \to \mathbb{R}^N$. Thus u is a singular point of $\mathcal{N}$ if and only if this map is not injective. We have the diagram of figure 17, with f replaced by q^* and f_N replaced by q^*_N.

Lemma 1. *Suppose that any d of the toric hyperkähler generators $\nu_1, \ldots, \nu_n$ are linearly independent over $\mathbb{R}$, i.e. all $d \times d$ minor determinants of the matrix G are non-vanishing. Then for any point $u \in \mathcal{N} - \{0\}$, the set $V(u)$ has at most $d-1$ elements (hence the set $N(u)$ has at least $r+1$ elements).*

Proof. Write $u_k = w_k^{(+)} + \mathbf{j} w_k^{(-)}$. As explained in section 5, the fact that u belongs to $\mathcal{N}$ means:

$$|w_k^{(+)}|^2 - |w_k^{(-)}|^2 = 2\nu_k \cdot a\,, \qquad w_k^{(+)} w_k^{(-)} = \nu_k \cdot b \tag{B.2}$$

for some $a \in \mathbb{R}^2$ and $b \in \mathbb{C}^2$. If k is an element of V, then $w_k^{(+)} = w_k^{(-)} = 0$ and we obtain $\nu_k \cdot a = 0$ and $\nu_k \cdot b = 0$. If V has more than $d-1$ elements, this implies $a = b = 0$ and thus $u = 0$ by eqs (B.2), since in this case the vectors ν_k $(k \in V)$ generate $\mathbb{R}^d$ by the assumption of the Lemma. Hence V has at most $d-1$ elements provided that $u \neq 0$.

Proposition 7. *The following are equivalent:*

(a) All $r \times r$ minor determinants of Q are nonzero

(b) Any d of the vectors $\nu_1, \ldots, \nu_n$ are linearly independent over $\mathbb{R}$, i.e. all $d \times d$ minor determinants of G are nonzero.

In this case, the origin of $\mathbf{H}^n$ is the only singular point of $\mathcal{N}$. A toric hyperkähler cone is called good *if it satisfies condition (a) (and thus (b)).*

Proof. Consider the diagram in figure 17 with f replaced by q^*. It is clear that conditions (a), (b) amount respectively to injectivity of $f = q_N^*$ or of g_V for all partitions $\{1, \ldots, n\} = V \cup N$ with $|V| = d$ and $|N| = r$. Hence equivalence of (a) and (b) follows immediately from Proposition 6.

Let us now assume that (a) (and thus (b)) holds. Fix a point $u \in \mathcal{N} - \{0\}$. As mentioned above, u is singular in $\mathcal{N}$ if and only if the map $q_N^* : \mathbb{R}^r \to \mathbb{R}^n$ (where $N = N(u)$) is not injective. This amounts to requiring that all maximal minor determinants of the matrix Q_N (obtained from Q by deleting the columns associated with $V(u)$ and keeping the columns associated with $N(u)$) vanish. Since $u \neq 0$ and (b) holds, Lemma 1 assures us that $N(u)$ has at least $r+1$ elements, so that Q_N has r rows and at least $r+1$ columns. This means that its maximal minors coincide with certain $r \times r$ minors of Q, whose determinant cannot vanish since (a) holds. Therefore, u cannot be a singular point of $\mathcal{N}$.

Corollary 1. *Let $X = \mathbf{H}^n///_0 T^{n-2}$ be an* eight-dimensional *toric hyperkähler cone. Then the following are equivalent:*

(a) The cone X is good.

(b) No two of the three-dimensional flats $H_1, \ldots, H_n$ coincide in $\mathbb{R}^6$.

(c) No two of the lines $h_1, \ldots, h_n$ coincide in $\mathbb{R}^2$.

Proof. For X an eight-dimensional toric hyperkähler cone, the hyperplanes h_j are lines in $\mathbb{R}^2$ which pass through the origin. Two such lines intersect outside of the origin if and only if they coincide. Since $H_j = h_j \times h_j \times h_j$, it follows that two flats can intersect outside of the origin if and only if they coincide. Now $h_j = \{a \in \mathbb{R}^2 | a \cdot \nu_j = 0\}$, so two lines h_i and h_j coincide if and only if ν_i and ν_j are linearly dependent. Since $d = 2$, the conclusion follows from Proposition 7.

B.2 Singularities of good toric hyperkähler cones

Let X be a good, d-dimensional toric hyperkähler cone. Since $\mathcal{N}$ is smooth except at the origin, all singularities of $X = \mathcal{N}/T^r$ outside its apex arise from points $u \in \mathcal{N} - \{0\}$ which have nontrivial finite stabilizers with respect to the T^r action. Since we assume this action to be effective (as follows from the fact that $\mathrm{coker}(q^*)$ is torsion-free), this can only happen if some of the coordinates of u vanish.[23] Considering the sets $N(u)$ and $V(u)$ as above, the torus T^n decomposes as $T^V \times T^N$, with T^V acting (trivially) on the vanishing coordinates and T^N acting on the nonvanishing components of u. It is clear that the T^r-stabilizer $\Gamma_u = Stab_{T^r}(u)$ of u coincides with the kernel of the map $T^r \to T^N$ induced from $T^r \to T^n$ by composing with the projection $T^n \to T^N$. Since X is good and $u \neq 0$, we have $|N| \geq r+1$ and q^*_N is injective. Hence the situation is precisely that considered in figure 17, with f replaced by q^*. Now Proposition 5 shows that $\Gamma_u = \mathrm{Tor}^{\mathbb{Z}}(A, \mathrm{U}(1))$, where $A = \mathrm{coker}(q^*_N)$. Since q^*_N is injective, Proposition 6 allows us to also compute A as the cokernel of the map $g_V : \mathbb{Z}^V \to \mathbb{Z}^d$.

In the case $d = 2$, Lemma 1 shows that at most one quaternion coordinate u_j can vanish unless u coincides with the apex of X. It follows that singularities of X can occur only along the loci $X_j = \{u \in X | u_j = 0\}$ discussed in section 5. For $u \in X_j - \{0\}$, one has $V(u) = \{j\}$ and the observation following Proposition 6 shows that Γ_u must be a cyclic group, determined by the greatest common divisor of the two components of the toric hyperkähler generator ν_j. This gives the following:

Proposition 8. *Let $X = \mathbf{H}^n///_0\, \mathrm{U}(1)^{n-2}$ be a good toric hyperkähler cone of real dimension eight. Then:*

1. *All singularities of X lie in one of the four-dimensional loci $X_j = \{u \in X | u_j = 0\}$. Two such loci intersect at precisely one point, namely the apex of X.*

2. *The locus $X_j - \{0\}$ is smooth if and only if the associated toric hyperkähler generator $\nu_j \in \mathbb{Z}^2$ is a primitive vector.*

3. *If ν_j is not primitive, then each point on the locus $X_j - \{0\}$ is a $\mathbb{Z}_m$ quotient singularity of X, where m is the greatest common divisor of the coordinates ν_j^1 and ν_j^2 of ν_j. The action of the generator of $\mathbb{Z}_{m_j}$ on the quaternion coordinate u_j transverse to the singular locus X_j is given by:*

$$u_j \to e^{\frac{2\pi i}{m_j}} u_j\,. \tag{B.3}$$

Recall from appendix A that the singularity group Γ_j along $X_j - \{0\}$ is given by the multiplicative group of solutions to the following system:

$$\prod_{\alpha=1}^{n-2} \lambda^{q_k^{(\alpha)}} = 1 \qquad \text{for } k \neq j\,. \tag{B.4}$$

[23] Indeed, the map of tori $T^r \to T^n$ is injective.

As explained in appendix A, any solution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-2})$ of this system will automatically satisfy the equation:

$$\prod_{\alpha=1}^{n-2} \lambda^{q_j^{(\alpha)}} = e^{2\pi i \frac{s}{m_j}} \qquad \text{for } k \neq j\,, \tag{B.5}$$

for some element $s = s_{\boldsymbol{\lambda}} \in \mathbb{Z}_{m_j}$, which is uniquely determined by $\boldsymbol{\lambda}$. The isomorphism $\Gamma_j \to \mathbb{Z}_{m_j}$ is given by the map $\boldsymbol{\lambda} \to s_{\boldsymbol{\lambda}}$.

Observation. If the toric hyperkähler cone is not good, then some of its singularities outside the apex are induced from singularities of the zero level set $\mathcal{N} = \vec{\mu}^{-1}(0)$. Such cases can be analyzed by algebraic geometry methods, upon using the toric embedding of X given in subsection 5.1.

C. Criterion for effectiveness of the projectivising $\mathrm{U}(1)$ action

Consider a d-dimensional toric hyperkähler cone $X = \mathbf{H}^n///_0\, \mathrm{U}(1)^r$ $(d = n-r)$. Let Q^{ismith} be the integral Smith form of the $r \times n$ charge matrix Q and $U \in \mathrm{SL}(r, \mathbb{Z})$, $V \in \mathrm{SL}(n, \mathbb{Z})$ such that $Q^{\mathrm{ismith}} = U^{-1}QV$. Since Q has trivial invariant factors, the matrix Q_{ismith} has the form $[I_r, 0]$, where I_r is the $r \times r$ identity matrix. Remember that X carries an action of $\mathrm{Sp}(1)$ induced by:

$$(u_1, \ldots, u_n) \to (u_1 t^{-1}, \ldots, u_n t^{-1})\,, \tag{C.1}$$

where t is a unit norm quaternion (viewed as an element of $\mathrm{Sp}(1)$). Consider the integral n-vector $F := V^t \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$ (i.e. F is the sum of all rows of V).

Proposition 9. *The $\mathbb{Z}_2$ subgroup $\{1, -1\}$ of $\mathrm{Sp}(1)$ acts trivially on X if and only if the components $F_{r+1}, \ldots, F_n$ are even. Equivalently, $\{-1, 1\}$ acts trivially on X if and only if there exist $1 \leq m \leq r$ and $1 \leq \alpha_1 < \alpha_2 < \cdots \alpha_m \leq r$ such that all components of the n-vector w defined as the sum of the rows $\alpha_1, \ldots, \alpha_m$ of Q are odd. In this case, the indices α_k with this property are uniquely determined, and the the action of the $\mathbb{Z}_2$ subgroup of $\mathrm{Sp}(1)$ on $\mathbf{H}^n$ coincides with the action on $\mathbf{H}^n$ of the $\mathbb{Z}_2$ subgroup of T^r generated by:*

$$\begin{aligned} \lambda_\alpha &= +1, \qquad \textit{for } \alpha \neq \alpha_j \\ \lambda_{\alpha_j} &= -1 \qquad \textit{for all } j = 1, \ldots, m\,. \end{aligned} \tag{C.2}$$

Proof. The generator of the given $\mathbb{Z}_2$ subgroup of $\mathrm{Sp}(1)$ acts on quaternion coordinates through sign inversion, $u_j \to -u_j$. This descends to a trivial action on X if and only if this transformation can be realized through the action of T^r, i.e. precisely when the following system admits a solution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \mathrm{U}(1)^r$:

$$\prod_{\alpha=1}^{r} \lambda_\alpha^{q_j^{(\alpha)}} = -1 \;\; \text{for all } j = 1, \ldots, n\,. \tag{C.3}$$

Upon writing $\lambda_\alpha = e^{2\pi i \phi_\alpha}$ (with $\phi_\alpha \in \mathbb{R}/\mathbb{Z}$), we can express (C.3) in the equivalent form:

$$Q^t\phi = \frac{1}{2}E \qquad (\mathrm{mod}\ (\mathbb{Z}^n))\,, \tag{C.4}$$

where $\phi = \begin{bmatrix}\phi_1\\ \phi_2\\ \dots\\ \phi_r\end{bmatrix}$ and E is the n-vector $\begin{bmatrix}1\\ 1\\ \dots\\ 1\end{bmatrix}$. Since $Q = UQ^{\mathrm{ismith}}V^{-1}$, this becomes:

$$(Q^{\mathrm{ismith}})^t\psi = F \quad (\mathrm{mod}\ (2\mathbb{Z})^n)\,, \tag{C.5}$$

where $\psi = 2U^t\phi \in (\mathbb{R}/(2\mathbb{Z}))^n$. Using the form $Q^{\mathrm{ismith}} = [I_r, 0]$, this gives $\psi = \begin{bmatrix}F_1\\ F_2\\ \dots\\ F_r\end{bmatrix} \in \mathbb{Z}^r/(2\mathbb{Z})^r = \mathbb{Z}_2^r$ and:

$$\begin{bmatrix}F_{r+1}\\ F_{r+2}\\ \dots\\ F_n\end{bmatrix} = 0 \quad (\mathrm{mod}\ (2\mathbb{Z})^d)\,, \tag{C.6}$$

which proves the first part of the proposition. Note that we have $\phi = \frac{1}{2}U^{-t}\psi \in (\frac{1}{2}\mathbb{Z})^r/\mathbb{Z}^r \approx (\mathbb{Z}_2)^r$, if we identify the subgroup $\{0, 1/2\}$ of $\mathbb{Q}/\mathbb{Z}$ with $\mathbb{Z}_2$. This means that the solution $\boldsymbol{\lambda}$ of (C.3) (when it exists) has the form $\lambda_\alpha = (-1)^{\sum_{\beta=1}^r (U^{-1})_{\beta\alpha}F_\beta}$, i.e. λ_α must be $+1$ or -1. Using this fact directly in (C.3), we see that a solution exists if and only if there exists a subcollection $\alpha_1, \dots, \alpha_m$ of the rows of Q whose sum w is a row vector all of whose entries are odd (α_j are those indices α for which $\lambda_{\alpha_j} = -1$). To see why the rows $\alpha_1, \dots, \alpha_m$ are uniquely determined, suppose that there exists another choice $\alpha'_1, \dots, \alpha'_{m'}$ of rows with this property, and let w' be the integral n-vector obtained by adding all these rows (by assumption, both w and w' have only odd entries). Eliminating the common rows between these two collections, we obtain two disjoint sets S_1 and S_2 of rows of Q which have the property that all entries of the vector $\sum_{k\in S_1} row(Q,k) - \sum_{k\in S_2} row(Q,k) = \sum_{j=1}^m row(Q,\alpha_j) - \sum_{j=1}^{m'} row(Q,\alpha'_j) = w - w'$ are even; but this immediately implies that all $r\times r$ minor determinants of Q are even, so that the discriminantal divisor $\mathfrak{g}(Q)$ would be even. This contradicts the assumption $\mathfrak{g}(Q) = 1 \Leftrightarrow Q^{\mathrm{ismith}} = [I_r, 0]$ made in the definition of toric hyperkähler spaces. The second part of the proposition follows.

D. Singularities of the twistor space along Y_e

Let Y be the twistor space Y associated with a good, eight-dimensional toric hyperkähler cone. Consider the locus $Y_e = Y_{\epsilon(e)}$ associated with an edge e of the characteristic polygon Δ. Remember that $Y_e = X_e/T^1$, where $X_e = \vec{\pi}^{-1}(e \times \{0_{\mathbb{R}^2}\} \times \{0_{\mathbb{R}^2}\})$ is a T^2 fibration over $e \subset \mathbb{R}^2$. We want to describe the quotient of the T^2 fiber of X_e by T^1. For this, we shall use the fact that the T^2 fiber itself is a quotient of T^n by $T^r = T^{n-2}$.

To describe this in terms of lattices, consider the decomposition $\mathbb{Z}^{n-1} = \mathbb{Z}^{n-2} \times \mathbb{Z}$ associated to the product $T^{n-1} = T^{n-2} \times T^1$ and the corresponding projections $p_1 : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-2}$, $p_2 : \mathbb{Z}^{n-1} \to \mathbb{Z}$ and injections $j_1 : \mathbb{Z}^{n-2} \to \mathbb{Z}^{n-1}$, $j_2 : \mathbb{Z} \to \mathbb{Z}^{n-1}$.

We have a split exact sequence:

$$0 \longrightarrow \mathbb{Z}^{n-2} \xrightarrow{j_1} \mathbb{Z}^{n-1} \xrightarrow{p_2} \mathbb{Z} \longrightarrow 0\,. \tag{D.1}$$

Restricting to the locus X_e, we are left with the non-vanishing variables $w_j^{(\epsilon_j(e))}$, and with the exact sequence (7.18) of subsection 7.5:

$$0 \longrightarrow \mathbb{Z}^{n-2} \xrightarrow{q_e^*} \mathbb{Z}^n \xrightarrow{g_e} \mathbb{Z}^2 \longrightarrow 0\,. \tag{D.2}$$

As in subsection 7.5, the middle term of (D.2) corresponds to the T^n action obtained by restricting the 'toric' diagonal T^{2n} action:

$$w_j^{(+)} \to \Lambda_j w_j^{(+)}\,, \qquad w_j^{(-)} \to \Lambda_{j+n} w_j^{(-)} \tag{D.3}$$

to the non-vanishing coordinates $w_j^{(\epsilon_j(e))}$:

$$w_j^{(\epsilon_j(e))} \to \lambda_j w_j^{(\epsilon_j(e))}\,, \tag{D.4}$$

where $\lambda_j = \Lambda_{j+(1-\epsilon_j(e))\frac{n}{2}}$.

The projectivising U(1) action on $w_j^{(\epsilon_j)}$ results from the map $\gamma : \mathbb{Z} \to \mathbb{Z}^n$ defined through:

$$\gamma(1) = t := \begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}. \tag{D.5}$$

The action of $T^{n-1} = T^{n-2} \times T^1$ results from the map $\bar{q}_e^* = q_e^* \circ p_1 + \gamma \circ p_2 : \mathbb{Z}^{n-1} \to \mathbb{Z}^n$. If $u = (v, m)$ is an element of $\mathbb{Z}^{n-1} = \mathbb{Z}^{n-2} \times \mathbb{Z}$ (with $v \in \mathbb{Z}^{n-2}$ and $m \in \mathbb{Z}$), then $\bar{q}_e^*(u) = q_e^*(v) + \gamma(m)$. The $(n-1) \times n$ matrix of the transpose map $\bar{q}_e$ is given by $\bar{Q}_e = \begin{bmatrix} Q_e \\ 1 \dots 1 \end{bmatrix}$.

Observation. The matrix $\bar{Q}_e$ is related to the toric approach to the singularities of Y outlined in subsection 7.9. Indeed, $\bar{Q}_e$ is obtained from the toric charge matrix $\tilde{Q} = \begin{bmatrix} Q, -Q \\ 1 \dots 1 \end{bmatrix}$ of the ambient space $\mathbb{T}$ upon keeping only those columns associated with the coordinates $w_j^{(\epsilon_j(e))}$ which do not vanish on the locus Y_e. From this perspective, the singularity type along Y_e can be determined by using the results of appendix A, i.e. by determining the integral Smith form of the matrix $\bar{Q}_e$. In this approach, it is not clear that the singularity group along Y_e is always a cyclic group. This facts only becomes clear when considering the alternate description used in subsection 7.5. The purpose of this appendix is to explain the equivalence of these two methods, and to provide a rigorous justification of the latter.

The injection γ induces a map $\alpha = g_e \circ \gamma : \mathbb{Z} \to \mathbb{Z}^2$, which describes the embedding of the projectivising U(1) in the T^2 fiber of X_e. This map is specified by:

$$\alpha(1) = g_e(t) = \sum_{j=1}^{n} \epsilon_j(e) \nu_j = \nu_e\,. \tag{D.6}$$

Lemma 2. *(a) The vector ν_e does not vanish. Therefore, the map α_e is nonzero (and injective).*

(b) The map $\bar{q}_e^*$ is injective, i.e. the matrix $\bar{Q}_e$ has maximal rank.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathbb{Z}^{n-2} \arrow[d, "q_e^*"] & & \\
0 \arrow[r] & \mathbb{Z}^{n-1} \arrow[ur, "p_1"] \arrow[r, "\bar{q}_e^*"] \arrow[d, shift left, "p_2"] & \mathbb{Z}^n \arrow[r, "\bar{g}_e"] \arrow[d, "g_e"] & A_e \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[u, shift left, "j_2"] \arrow[ur, "\gamma"] \arrow[r, "\alpha"] & \mathbb{Z}^2 \arrow[r, "\beta"] \arrow[d] & A_e' \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

Figure 19: Lattice diagram for the embedding of T^1 into the T^2 fiber on the locus X_ϵ.

Proof. To show (a), consider the vector p_e (associated with the middle point of the edge e) as in subsection 7.5. Since $\epsilon_j(e) = \mathrm{sign}(p_e \cdot \nu_j)$, we have:

$$p_e \cdot \nu_e = \sum_{j=1}^{n} \epsilon_j(e) p_e \cdot \nu_j = \sum_{j=1}^{n} |p_e \cdot \nu_j| \,. \tag{D.7}$$

Therefore, vanishing of ν_e would imply $p_e \cdot \nu_j = 0$ i.e. $p_e \in D_j$ for all j (j are the principal diagonals of the characteristic polygon Δ). This is impossible, since p_e belongs to the interior of the edge e.

To show (b), consider an element $u \in \mathbb{Z}^{n-1}$ such that $\bar{q}_e^*(u) = 0$. Writing $u = (v, m)$ with $v \in \mathbb{Z}^{n-2}$ and $m \in \mathbb{Z}$, we have $0 = \bar{q}_e^*(u) = q_e^*(v) + \gamma(m)$, so that $\alpha(m) = g_e(\gamma(m)) = -g_e(q_e^*(v)) = 0$ since $g_e \circ q_e^*$ vanishes. Since α is injective, this gives $m = 0$ and $q_e^*(v) = 0$. This implies $v = 0$ since q_e^* is injective. Thus $u = 0$ and $\bar{q}_e^*$ is injective.

In view of the Lemma, one has short exact sequences:

$$0 \longrightarrow \mathbb{Z}^{n-1} \stackrel{\bar{q}_e^*}{\longrightarrow} \mathbb{Z}^n \stackrel{\bar{g}_e}{\longrightarrow} A_e \longrightarrow 0 \,. \tag{D.8}$$

and

$$0 \longrightarrow \mathbb{Z} \stackrel{\alpha}{\longrightarrow} \mathbb{Z}^2 \stackrel{\beta}{\longrightarrow} A_e' \longrightarrow 0 \,. \tag{D.9}$$

Both A_e and A_e' may contain torsion; this corresponds to possible fixed points of the T^{n-1} and T^1 actions on X_e. The situation is summarized in figure 19.

Proposition 10. *The square formed by the maps $\bar{q}_e^*, g_e, p_2$ and α commutes. Moreover, there exists a unique morphism $\Phi : A_e \to A_e'$ which produces a commutative square on the right, and this morphism is an isomorphism.*

Proof.

1. We first show that the square indicated commutes. If $u = (v, m)$ is an element of $\mathbb{Z}^{n-1} = \mathbb{Z}^{n-2} \times \mathbb{Z}$, then $p_1(u) = v$ and $p_2(u) = m$. We have $g_e \circ \bar{q}_e^*(u) = g_e(q_e^*(v) + \gamma(m)) = g_e \circ \gamma(m) = \alpha(m) = \alpha \circ p_2(u)$, where we used the definition of $\bar{g}_e$ and α as well as exactness of the vertical sequence. Thus $g_e \circ \bar{q}_e^* = \alpha \circ p_2$, which is the desired statement.

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$$\begin{tikzcd}
1 \arrow[r] & \Gamma_e \arrow[r] \arrow[d, "\Phi_*"'] & T^{n-1} \arrow[r, "\bar{q}_e^*"] \arrow[d, "p_2"'] & T^n \arrow[r, "\bar{g}_e"] \arrow[d, "g_e"'] & A_e \otimes_{\mathbb{Z}} U(1) \arrow[r] \arrow[d, "\Phi"'] & 1 \\
1 \arrow[r] & \Gamma'_e \arrow[r] & T^1 \arrow[r, "\alpha"] & T^2 \arrow[r, "\beta"] & A'_e \otimes_{\mathbb{Z}} U(1) \arrow[r] & 1
\end{tikzcd}$$

Figure 20: The associated diagram of tori.

2. Forgetting the morphism j_2, we have maps p_2 and g_e which close the first square. Since the horizontal sequences are short exact, applying the 3-lemma shows that there exists a unique morphism $\Phi : A_e \to A'_e$ which closes the last square to a commutative diagram. Surjectivity of Φ follows from the commutative square on the right (since β and g_e are surjective). To show injectivity of Φ, consider an element $x \in A_e$ such that $\Phi(x) = 0$. Then $x = \bar{g}_e(y)$ with $y \in \mathbb{Z}^n$ and $g_e(y) \in \ker\beta = im\alpha$. Thus $g_e(y) = \alpha(m)$ for some $m \in \mathbb{Z}$. Since $\alpha = g_e \circ \gamma$, this means that $y = \gamma(m) + w$, with w an element of $\ker g_e = imq_e^*$. Hence there exists a $v \in \mathbb{Z}^{n-2}$ such that $y = \gamma(m) + q_e^*(v) = \bar{q}_e^*(v, m)$. This implies $y \in im\bar{q}_e^* = \ker \bar{g}_e$, and thus $x = \bar{g}_e(y) = 0$.

Consider the horizontal sequences in figure 19. Tensoring with U(1) gives two exact sequences on four terms (the associated long exact Tor sequences collapse because $\mathbb{Z}^n$ and $\mathbb{Z}^2$ are flat.). The leftmost terms of the resulting sequences are $\Gamma_e = \mathrm{Tor}^{\mathbb{Z}}(A_e, \mathrm{U}(1))$ and $\Gamma'_e = \mathrm{Tor}^{\mathbb{Z}}(A'_e, \mathrm{U}(1))$, the kernels of the maps $T^{n-1} \to T^n$ and $T^1 \to T^2$ (figure 20).

Corollary 2. *The groups Γ_e and Γ'_e are isomorphic.*

Proof. Follows from naturality of Tor.

This shows that one obtains the same result by computing Γ_e from the maps $T^1 \to T^2$ or $T^{n-2} \to T^n$ on the locus Y_e.

It is now easy to describe the orbifold group of the twistor space along Y_e, as well as its action on the transverse coordinates $w_j^{(-\epsilon_j(e))}$. For this, we have to remember that the group which acts effectively on X is $\mathrm{U}(1)_{\mathrm{eff}} = \mathrm{U}(1)$ or $\mathrm{U}(1)/\mathbb{Z}_2$, according to the criterion of appendix C. This means that the singularity group coincides with Γ_e or $\Gamma_e/\mathbb{Z}_2$.

Corollary 3. *Assume that Y_e is nondegenerate. Then the group Γ_e is isomorphic with $\mathbb{Z}_{m_e}$, where m_e is the greatest common divisor of the coordinates of ν_e. The generator of $\mathbb{Z}_{m_e}$ acts on the coordinates $w_j^{(-\epsilon_j(e))}$ transverse to Y_e through:*

$$w_j^{(-\epsilon_j(e))} \to e^{\frac{2\pi i}{m_e}} \qquad w_j^{(-\epsilon_j(e))} . \tag{D.10}$$

If the projectivising U(1) *acts effectively on X, then the singularity group of Y along Y_e coincides with Γ_e, with the transverse action given above. Otherwise, the singularity group is $\Gamma_e/\mathbb{Z}_2$, with transverse action induced by (D.10).*

Proof. The sequence (D.9) presents $A'_e = A_e$ as:

$$A'_e = \mathbb{Z}^2/\alpha(\mathbb{Z}) = \mathbb{Z}^2/\mathbb{Z}\nu_e\,, \tag{D.11}$$

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and thus the group Γ_e as:

$$\Gamma_e = \mathrm{Tor}^{\mathbb{Z}}(A'_e, \mathrm{U}(1)) = T(A'_e) = \mathbb{Z}_{m_e} . \tag{D.12}$$

The embedding of Γ_e into the T^n factor of T^{2n} which acts diagonally on the transverse coordinates $w_j^{(-\epsilon_j(e))}$ is induced by a copy of the map γ given in (D.5). This takes the generator of $\mathbb{Z}_{m_e}$ into the element $\lambda = (e^{\frac{2\pi i}{m_e}}, \dots, e^{\frac{2\pi i}{m_e}})$. This immediately gives the transverse action. The remaining statements are obvious.

Observation 1. Recall that the torus T^n appearing in the presentation $X = \mathbf{H}^n///_0 T^n$ acts on $w_j^{(\pm)}$ as $w_j^{(\pm)} \to \Lambda_j^{\pm 1} w_j^{(\pm)}$. Therefore, the embedding of $\mathbb{Z}_{m_e}$ into this torus takes the generator of $\mathbb{Z}_{m_e}$ into $(e^{-\epsilon_1(e)\frac{2\pi i}{m_e}}, \dots, e^{-\epsilon_n(e)\frac{2\pi i}{m_e}})$.

Observation 2. When combined with the results of appendix A, the two Corollaries show that the integral Smith form of the matrix $\bar{Q}_e$ is always of the type $\bar{Q}_e^{\mathrm{ismith}} = [\mathrm{diag}(1, \dots, 1, m_e), 0]$, i.e. at most one invariant factor of $\bar{Q}_e$ can be nontrivial. This explains why the two methods for identifying the singularities of Y along Y_e lead to the same result.

E. Embedding X in a toric variety

Consider a toric hyperkähler cone described by the sequence (4.8). Recall the embedding $X \subset \mathbb{S}$, where $\mathbb{S} = \mathbb{C}^{2n}/(\mathbb{C}^*)^r$. The toric variety $\mathbb{S}$ results upon decomposing the quaternion coordinates into complex coordinates as in (5.2). Since $w_j^{(+)}$ and $w_j^{(-)}$ acquire opposite charges $q_j^{(\alpha)}$ and $-q_j^{(\alpha)}$, this is described by the map $j = (id, -id) : \mathbb{Z}^n \to \mathbb{Z}^{2n}$. The complex coordinates have been arranged as $z_1 = w_1^{(+)}, \dots, z_n = w_n^{(+)}, z_{n+1} = w_1^{(-)}, \dots, z_{2n} = w_n^{(-)}$, and are acted on by T^{2n} through the diagonal transformations $z_\rho \to \Lambda_\rho z_\rho$. The quotienting torus T^r maps to T^{2n} through the composite map $\hat{q}^* = j \circ q^* = (q^*, -q^*)$, which is obviously injective (since q^* is).

Proposition 11. *The cokernel of $\hat{q}^*$ is torsion free. In particular, we have an exact sequence:*

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\hat{q}^*} \mathbb{Z}^{2n} \xrightarrow{\hat{g}} \mathbb{Z}^{2d+r} \longrightarrow 0\,; \tag{E.1}$$

Proof. By the structure theorem of lattice maps, we have bases v_α of $\mathbb{Z}^r$ and u_j of $\mathbb{Z}^n$ such that $q^*(v_\alpha) = u_\alpha$ (remember that we assume $Q^{\mathrm{ismith}} = [I_r, 0]$). It is easy to check that the vectors $U_\alpha := (u_\alpha, -u_\alpha), U_{-\alpha} = (u_\alpha, 0)$ $(\alpha = 1, \dots, r)$ and $U_j := (u_j, 0)$, $U_{-j} = (0, u_j)$ $(j = r+1, \dots, n)$ form a basis of $\mathbb{Z}^{2n}$. Since $\hat{q}^*(v_\alpha) = U_\alpha$, the map $\hat{q}^*$ has integral Smith form with respect to the bases v and U, with unit torsion coefficients. This implies that its cokernel is torsion-free.

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